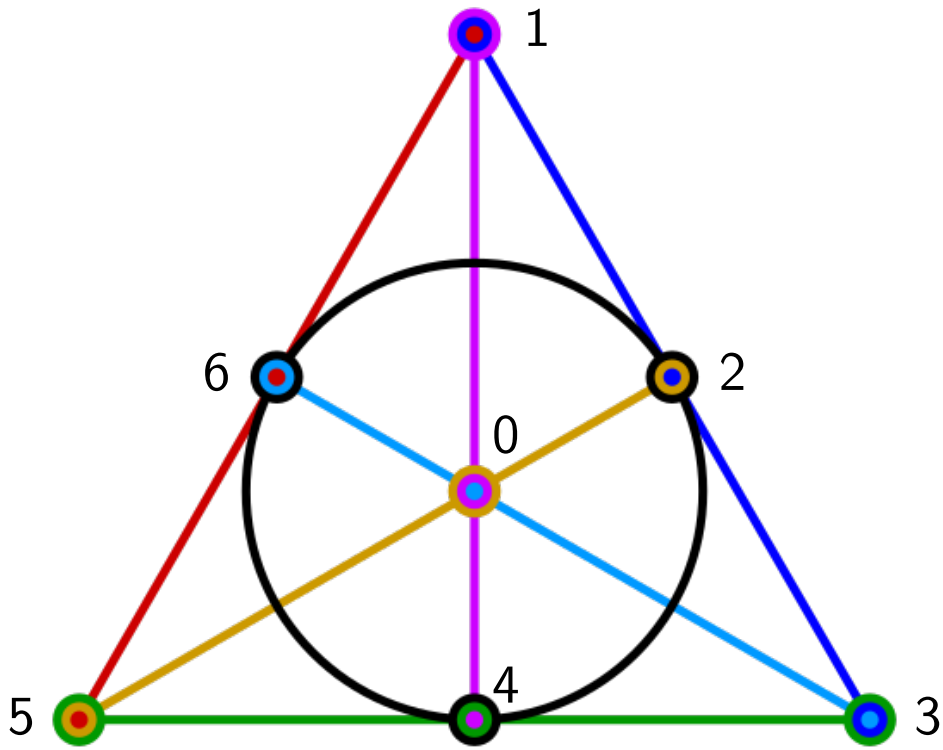
What are...Steiner systems?

Or: Finite geometry and puzzles

Finite projective planes (PP)

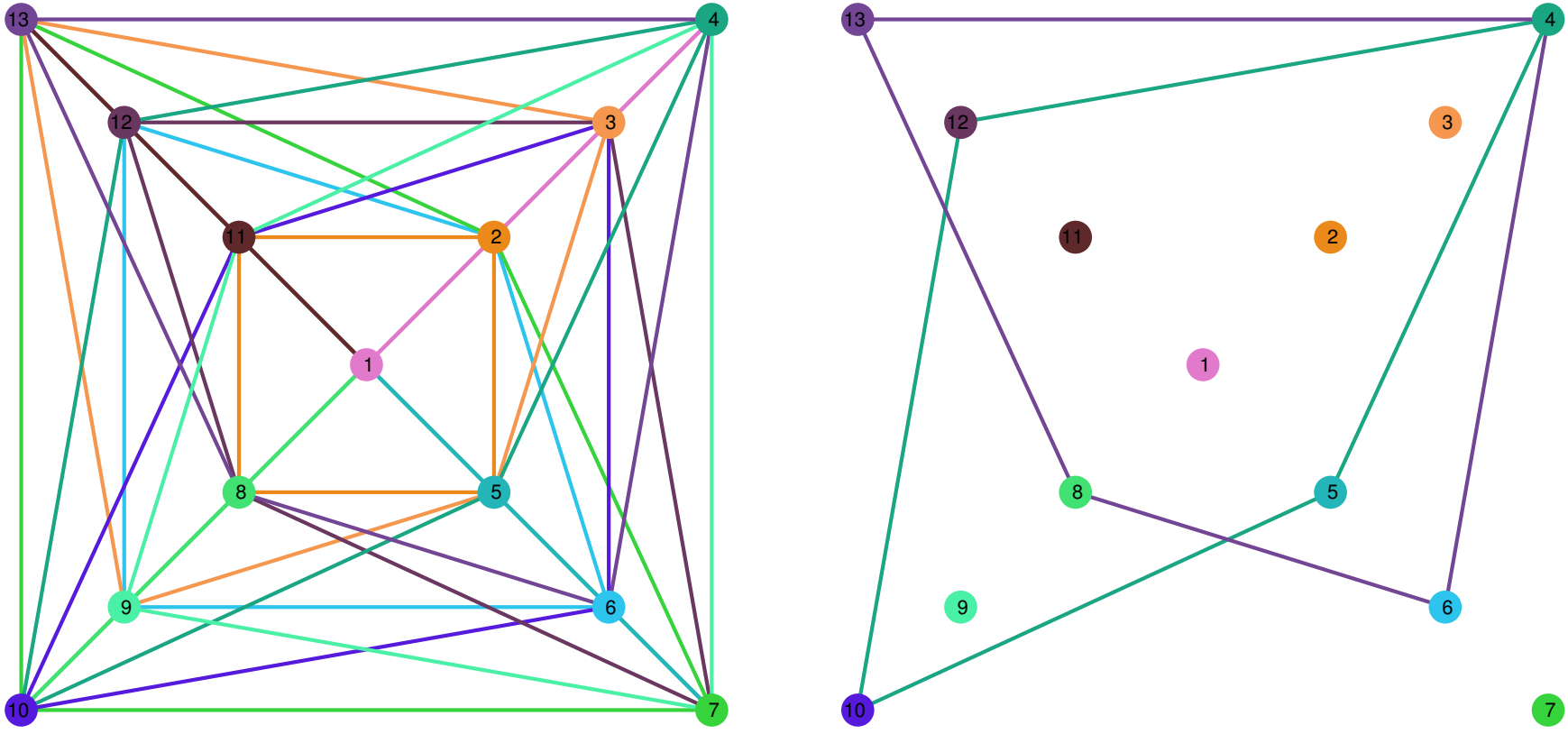
- For every two distinct points, there is exactly one line that contains both points
- There exists a set of four points, no three of which belong to the same line
- The intersection of any two distinct lines contains exactly one point



Fano plane $F = \{ 014, 025, 036, 123, 156, 246, 345 \}$ $S(2,3,7)$

Do PP of order N exist? Unclear...

PP(3), with 13 points and 13 polygonal "lines", passes the projective-plane test. PP(3), with 13 points and 13 polygonal "lines", passes the projective-plane test.



- PP have $N^2 + N + 1$ points and lines
- PP have $N + 1$ points on each line and lines through each point
- The intersection of any two distinct lines contains exactly one point
- Steiner systems $S(2, N + 1, N^2 + N + 1)$

Kirkman's school girl problem ∼1850

Fifteen young ladies in a school walk out three abreast for seven days in succession:

it is required to arrange them daily so that no two shall walk twice abreast

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
ABC	ADG	AEO	AIM	AFJ	AHK	ALN
DEF	BEH	BIJ	BDL	BKO	BGN	BFM
GHI	CJM	CDN	CEK	CGL	CFI	CHO
JKL	FKN	FHL	FGO	DHM	DJO	DIK
MNO	ILO	GKM	HJN	EIN	ELM	EGJ

A neutral formulation. Does there exist a Steiner system $S(2, 3, 15)$?

Enter, the definition/theorem

A Steiner system $S(t, k, n)$ with $t < k < n$ consists of:

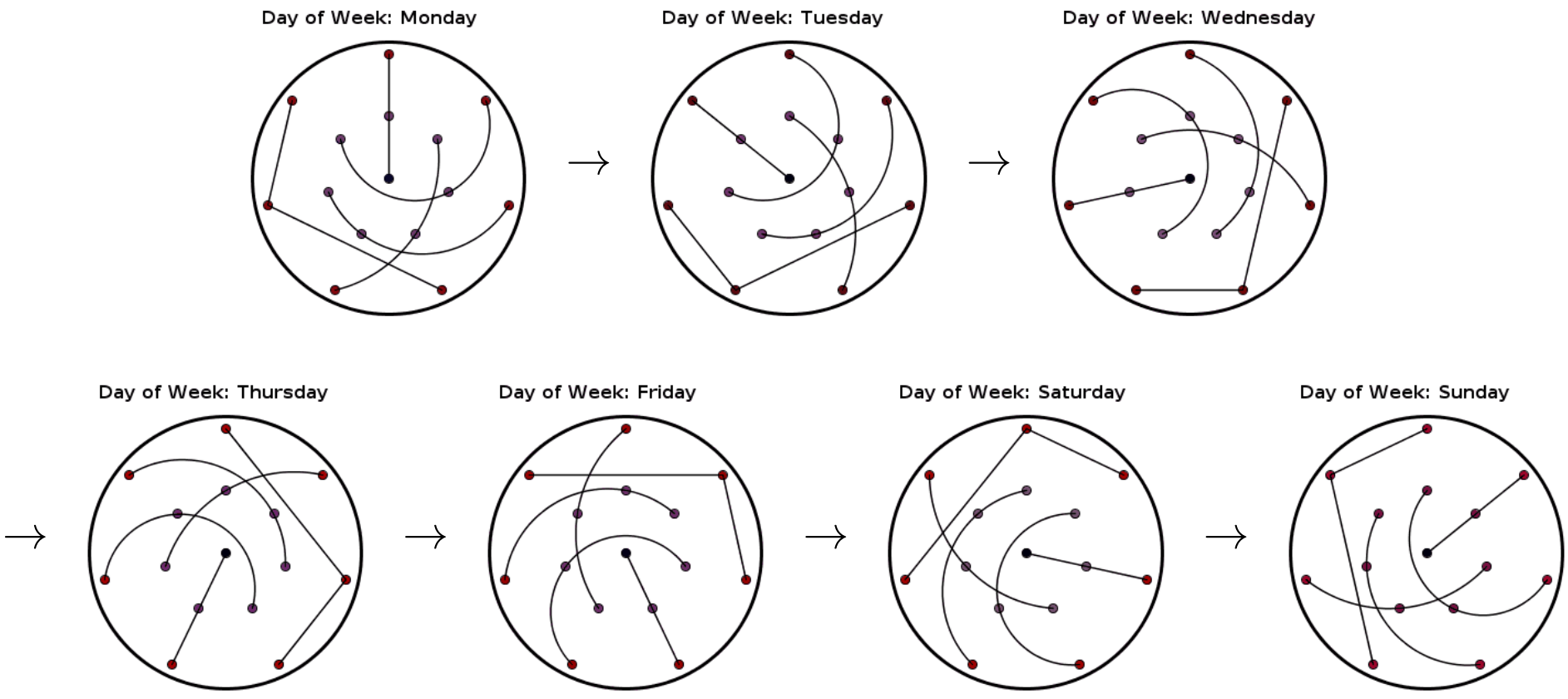
(a) An n-element set S n

(b) Blocks, k-element subsets of S k

(c) Every t-element subset of S is contained in exactly one block t

- Existence questions are usually very hard
- They exist for "big enough" t, but none are known for large t
- For big t these are very hard to construct but often have unexpected relations to other parts of mathematics
- For example, the Mathieu group M_{24} is the automorphism group of the unique system $S(5, 8, 24)$
- There are generalizations known as block designs
- There are many applications *e.g.* in statistics or code correction

$S(2, 3, 15)$ in action



- 15 points
- Lines containing 3 points
- Every 2 points are contained in exactly one line

Thank you for your attention!

I hope that was of some help.