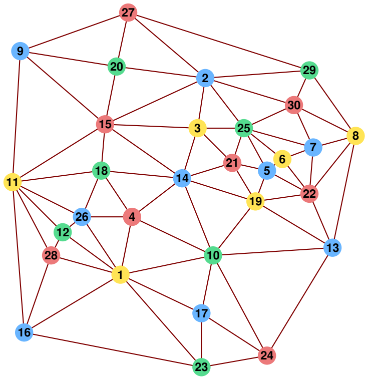
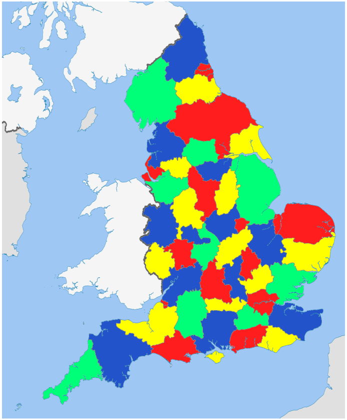
What is...Heawood's conjecture?

Or: I need more than four colors

A classic: the four color theorem

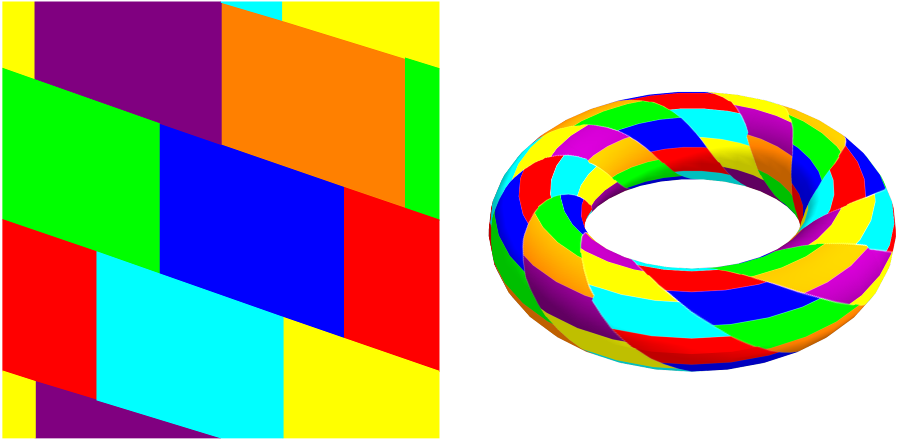




- Four colors suffice: every planar graph is four-colorable
- Conjectured by Francis Guthrie ∼1852 (counties of England)
- Open for more than 100 years; known proofs are complicated

The Heawood conjecture is a generalization with a much simpler proof Strange

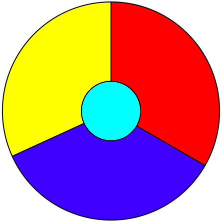
Seven instead of four



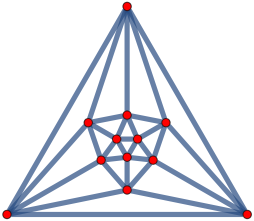
Heawood ∼1890. The same question on a torus needs 7 colors!

The sphere $g = 0$ vs. other surfaces $g > 0$

- Step ($g = 0$ easy, $g > 0$ hard). Find a lower bound by constructing a graph, *e.g.*



- Step ($g = 0$ too big, $g > 0$ works). Find an upper bound a Euler characteristic type argument, *e.g.*



Every vertex has degree at most 5

Remove such a vertex *v*

Inductively color the rest using 6 colors

Add *v* – there is at least one free color

Enter, the theorem

c colors suffice : every planar graph on a genus g surface is c-colorable where

$$c = \left\lfloor \frac{1}{2}\left(7 + \sqrt{1 + 48g}\right) \right\rfloor$$

- The sequence reads

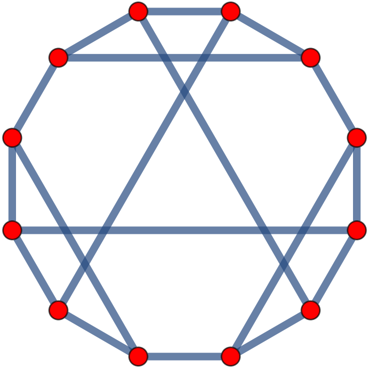
g	0	1	2	3	4	5	6	7	8	9	10
c	4	7	8	9	10	11	12	12	13	13	14

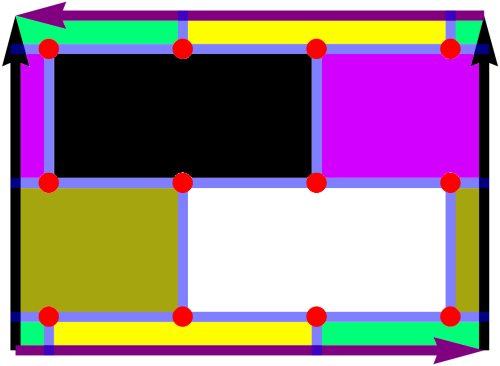
- Lower bound for $g = 0$ is clear; lower bound for $g > 0$ took a while
- Upper bound for $g = 0$ took a while; lower bound is due to Heawood
- If one replaces g by the Euler characteristic χ, then

$$c = \left\lfloor \frac{1}{2}\left(7 + \sqrt{49 - 24\chi}\right) \right\rfloor$$

  is the formula
- The above works for non-orientable surfaces except the Klein bottle

What about non-orientable surfaces?





For non-orientable surfaces the formula remains true except for the Klein bottle:

- Heawood's formula predicts seven colors
- The Franklin graph needs only six
- This is a funny small number coincidence

Thank you for your attention!

I hope that was of some help.