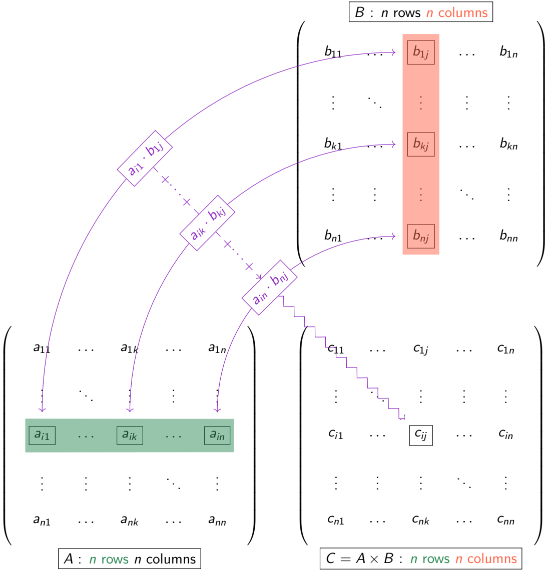
What is...the Strassen algorithm?

Or: Divide and conquer

A good old friend: matrix multiplication



n^3 multiplications – n rows, n columns and n claps. Can we do better?

7 **instead of** 8

Naively we need 8 calls:

M	1	2	3	4	5	6	7	8
Rule	$a_{11} \cdot b_{11}$	$a_{11} \cdot b_{12}$	$a_{12} \cdot b_{21}$	$a_{12} \cdot b_{22}$	$a_{21} \cdot b_{11}$	$a_{21} \cdot b_{12}$	$a_{22} \cdot b_{21}$	$a_{22} \cdot b_{22}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} M1 + M3 & M2 + M4 \\ M5 + M7 & M6 + M8 \end{pmatrix}$$

Strassen can do with 7 calls:

S	1	2	3	4
Rule	$(a_{11} + a_{22}) \cdot (b_{11} + b_{22})$	$(a_{21} + a_{22}) \cdot b_{11}$	$a_{11} \cdot (b_{12} - b_{22})$	$a_{22} \cdot (b_{21} - b_{11})$

5	6	7
$(a_{11} + a_{12}) \cdot b_{22}$	$(a_{21} - a_{11}) \cdot (b_{11} + b_{12})$	$(a_{12} - a_{22}) \cdot (b_{21} + b_{22})$

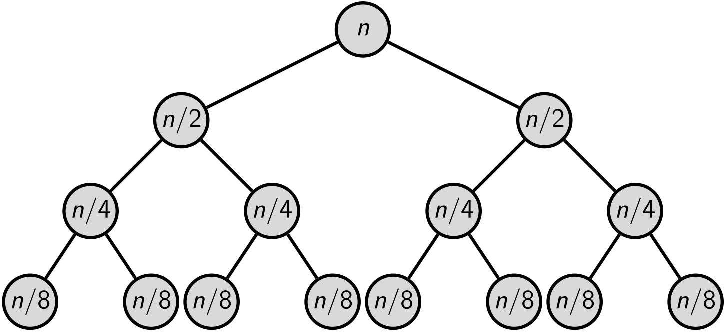
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} S1 + S4 - S5 + S7 & S3 + S5 \\ S2 + S4 & S1 - S2 + S3 + S6 \end{pmatrix}$$

Divide and conquer

- Break the matrices in blocks of size $n/2 \times n/2$ Divide!

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} S1 + S4 - S5 + S7 & S3 + S5 \\ S2 + S4 & S1 - S2 + S3 + S6 \end{pmatrix}$$

- Needs 7 calls on $n/2 \times n/2$ matrices
- Repeat recursively Conquer!



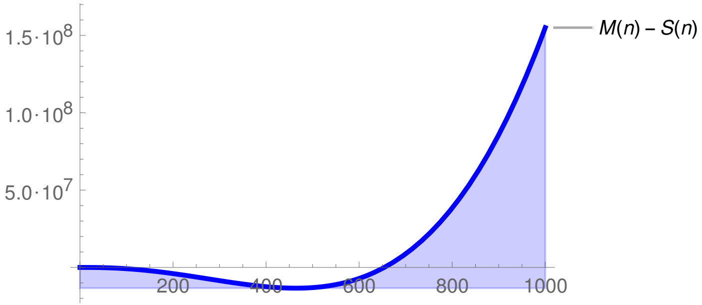
Strassen then needs $\approx n^{\log_2(7)}$ operations

Strassen's algorithm multiplies matrices with a cost of

$$S(n) = 7 \cdot 7^{\log_2(n)} - 6 \cdot 4^{\log_2(n)} \approx 7n^{\log_2(7)} - 6n^2 \approx n^{\log_2(7)} \text{ operations}$$

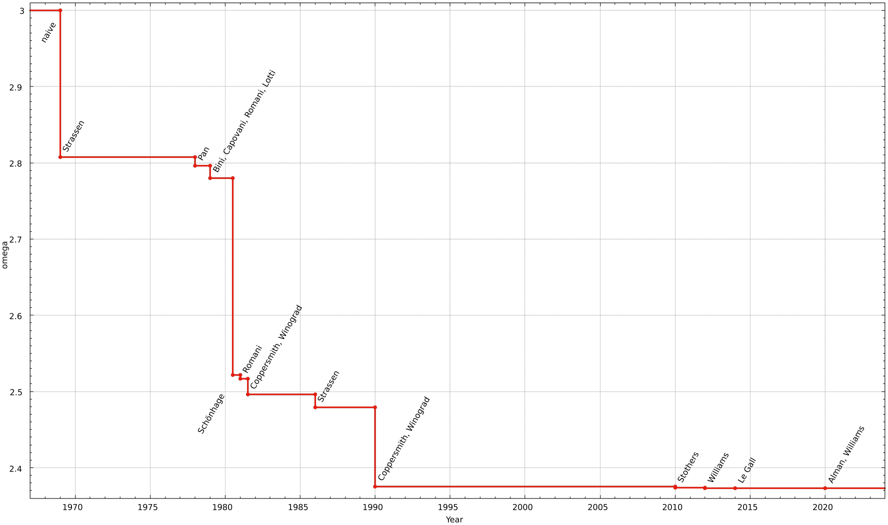
(a) Classical matrix multiplication needs $M(n) = 2n^3 - n^2 \approx n^3$ operations – n^3 multiplications and $n^3 - n^2$ additions

(b) $\log_2(7) \approx 2.807 \Rightarrow$ Strassen is much faster



(c) Strassen's ideas can be pushed further and modern algorithm are even faster

Can we get $\omega = 2$?



Matrix multiplication needs $n^\omega \log(n) \approx n^\omega$ operations, $2 \leq \omega < 2.4$ (bounds from 2020)

Thank you for your attention!

I hope that was of some help.