What is...the chaos game?

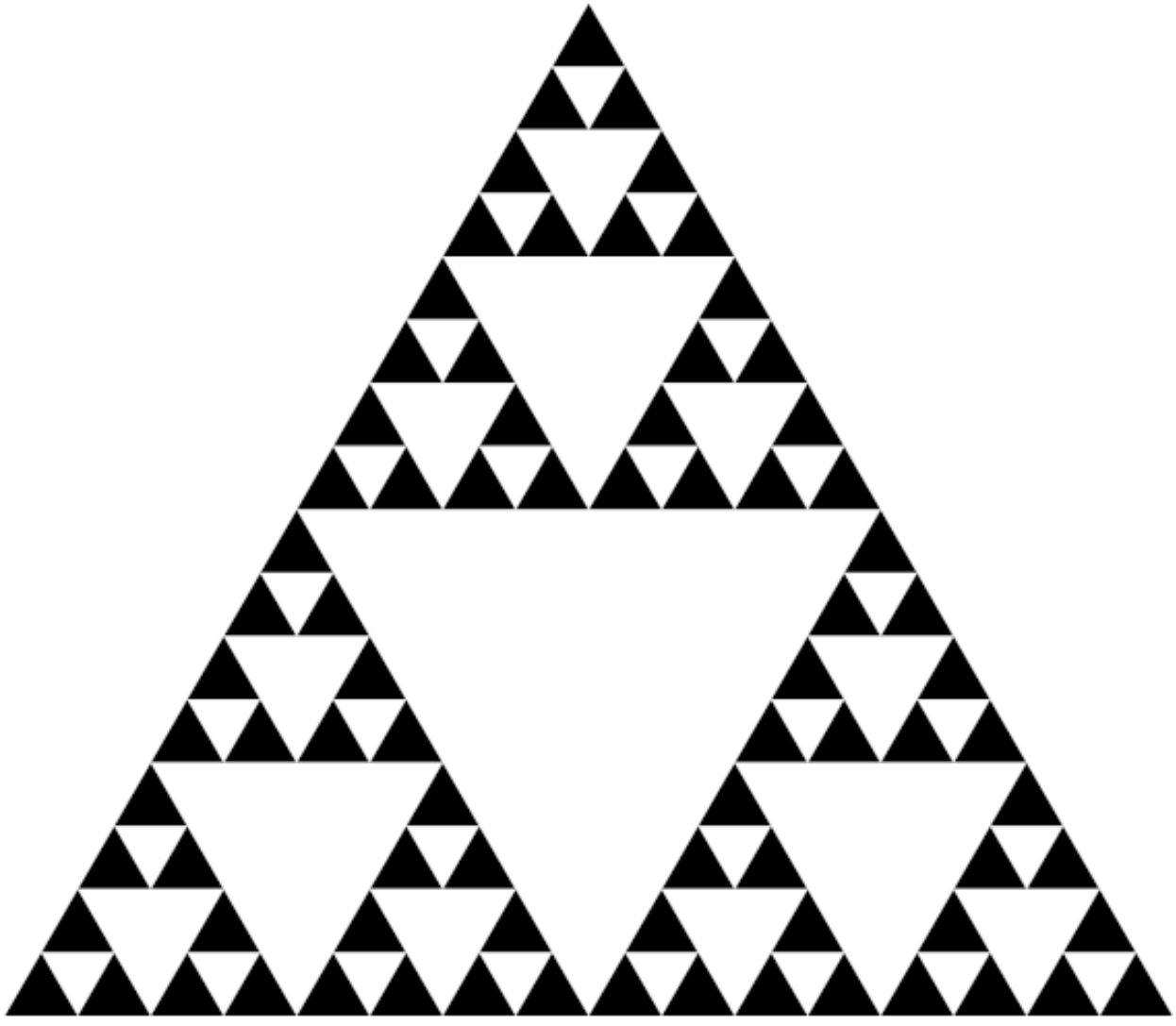
Or: Deterministic versus random

Fractals in stone (13th century, Rome)

Most commonly fractals are created by iterating a simple deterministic rule

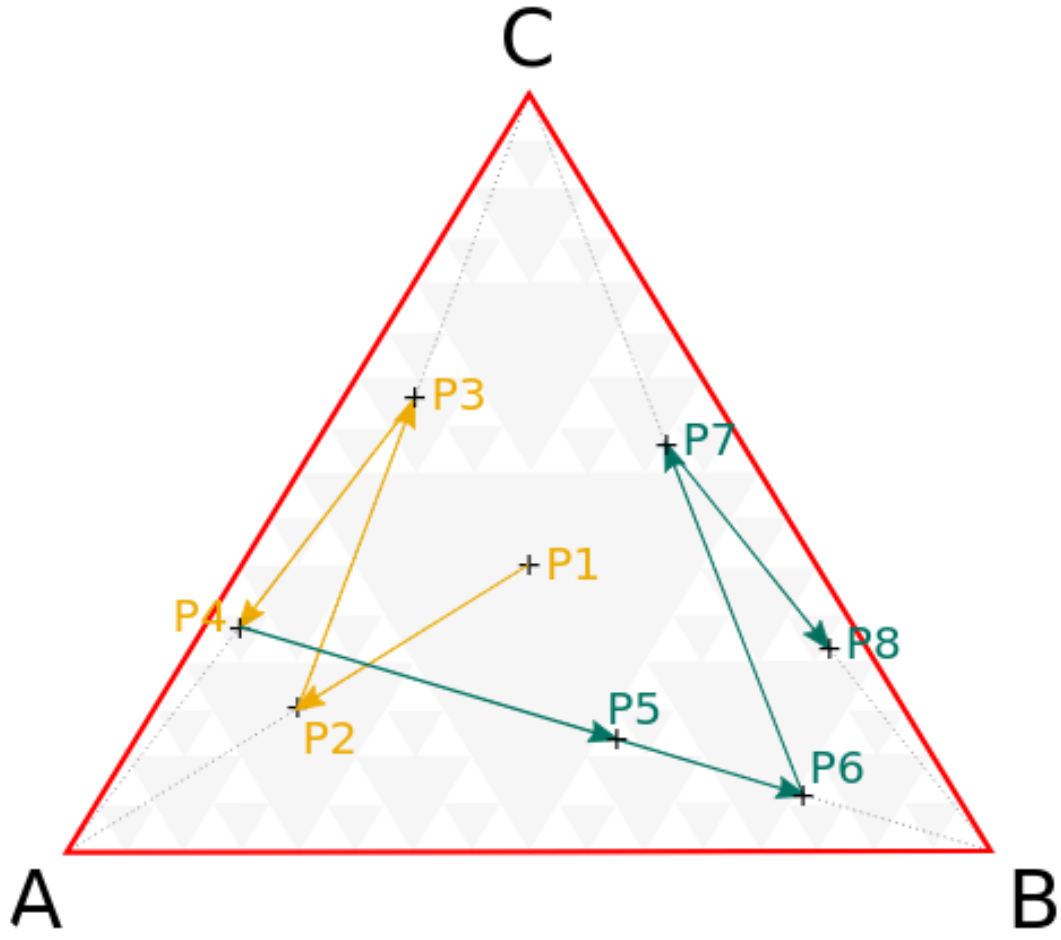
Question. How to create fractals using randomness ?

A classical way: Sierpinski's triangle using self-similarity



Creating fractals using self-similarity Deterministic

A random way: the chaos game



(a) Take a polygon *Pol*, choose $0 < r < 1$ and a point $p \in Pol$ Initialization

(b) Let p jump r steps in direction of a randomly chosen vertex of *Pol* and mark where p lands Randomize

(c) Repeat (b) Iteration

Enter, the theorem/philosophy!

Construct fractals via iterated function systems (IFS):

- Fix a finite set of contraction mappings

$$\{f_i : X \to X \mid i = 1, 2, ..., N\}$$

  on a complete metric space X
- A IFS has a unique nonempty compact fixed set S given by

$$S = \text{closure of } \bigcup_{i=1}^{N} f_i(S)$$

- Construction of S via the "doing all f_i at once" Deterministic
- Construction of S via the chaos game Random

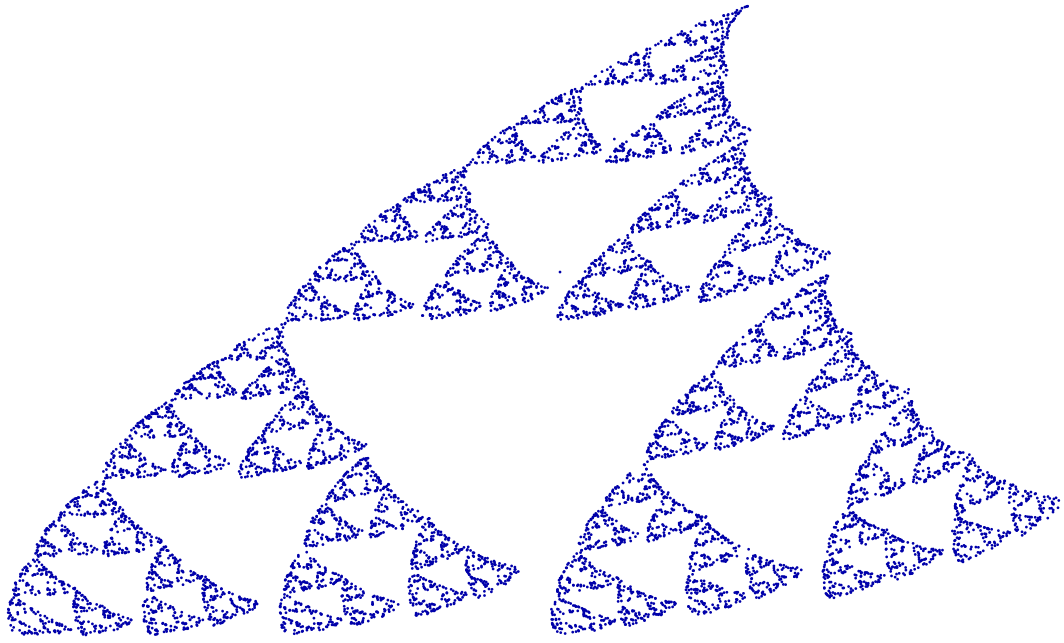
The proof uses a working horse: the Banach fixed-point theorem

A variation using affine transformations

$$\left(\begin{array}{cc|c} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5\sqrt{3} \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 0.5 & -0.1 & 1 \\ 0.1 & 0.5 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$



Same game as before with three fixed affine transformations which are then randomly chosen in each step

Thank you for your attention!

I hope that was of some help.