What is...hyperplane separation?

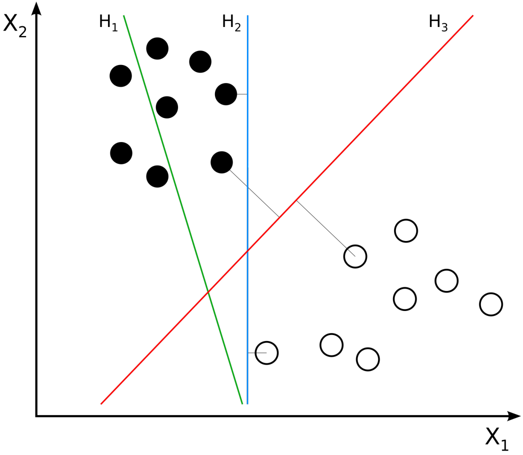
Or: Cutting data into bits

Data clusters

H_1 is bad

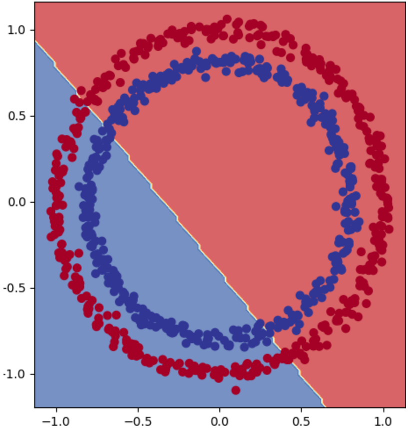
H_2 is ok

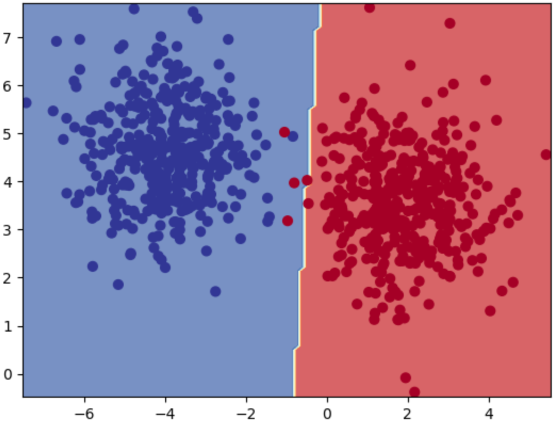
H_3 is optimal



- Say we want to separate data clusters
- Linear strategy Find a hyperplane separating the clusters
- For this video we ignore finding an optimal cutting hyperplane

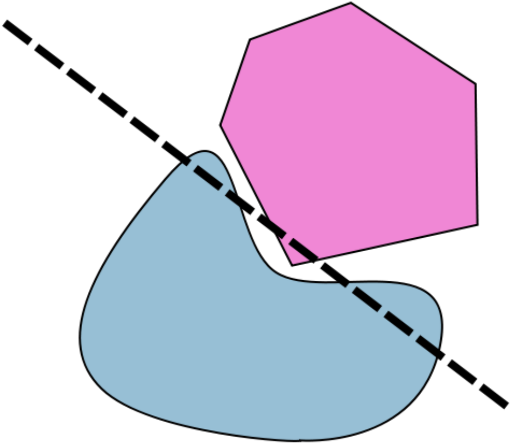
Ok, this sometimes fails





- Problem Circle clusters cannot be separated using hyperplanes
- Problem Overlapping (non-convex) clusters cannot be separated using hyperplanes
- Question In what generality can we separate using hyperplanes?

A more abstract setting

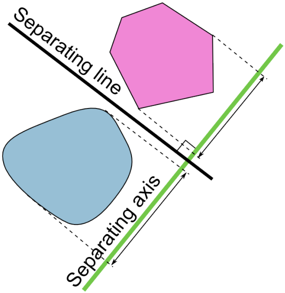


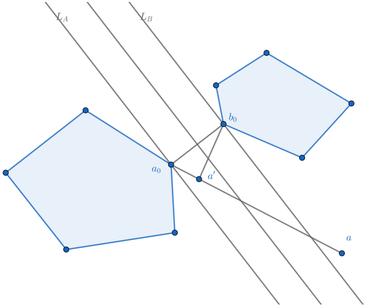
- The problem with our two subsets from before is that they are not convex
- Convex = it contains all line segments between its points
- Convexity makes sense in any real topological vector space

Enter, the theorem

$A, B \subset X$ nonempty and convex and disjoint, say A is compact and B closed, then:

A, B can be separated by a hyperplane





- In this version X only needs to be a real topological vector space
- A picture for $X = \mathbb{R}^2$ is above
- There are proofs that construct an "optimal" hyperplane

The Hahn–Banach theorem

Wikipedia on Hahn–Banach:

Continuous extension theorem [edit]

The Hahn–Banach theorem can be used to guarantee the existence of continuous linear extensions of continuous linear functionals.

> **Hahn-Banach continuous extension theorem**[14] — Every continuous linear functional f defined on a vector subspace M of a (real or complex) locally convex topological vector space X has a continuous linear extension F to all of X. If in addition X is a normed space, then this extension can be chosen so that its dual norm is equal to that of f.

In category-theoretic terms, the underlying field of the vector space is an injective object in the category of locally convex vector spaces.

On a normed (or seminormed) space, a linear extension F of a bounded linear functional f is said to be *norm-preserving* if it has the same dual norm as the original functional: $\|F\| = \|f\|$. Because of this terminology, the second part of the above theorem is sometimes referred to as the "norm-preserving" version of the Hahn–Banach theorem.[15] Explicitly:

> **Norm-preserving Hahn-Banach continuous extension theorem**[15] — Every continuous linear functional f defined on a vector subspace M of a (real or complex) normed space X has a continuous linear extension F to all of X that satisfies $\|f\| = \|F\|$.

- There are two versions of the Hahn–Banach theorem from functional analysis: analytic and geometric
- Analytic Bounded linear functionals defined on a vector subspace extend to the whole space
- Geometric Similar to the one from before
- Surprise They are equivalent

Thank you for your attention!

I hope that was of some help.