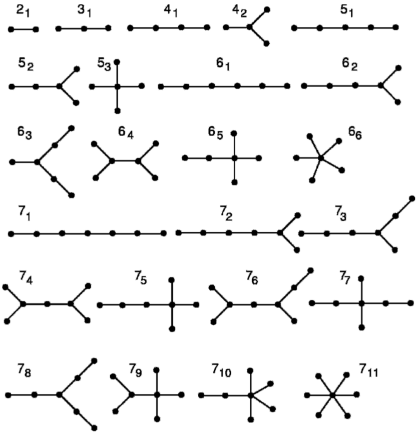
What is...the tree constant?

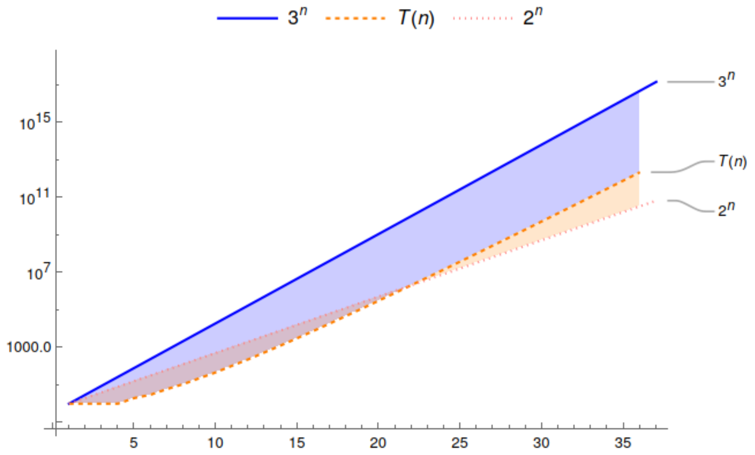
Or: Trees are difficult

Counting trees is difficult



- Tree = a graph with out nontrivial cycles
- Counting trees is very difficult (i.e. there is probably no nice formula giving the number of trees with n vertices $T(n)$)
- Task Find a way to count them, while not counting them!

What is the growth of $T(n)$?



- Ansatz $T(n) \sim s(n) \cdot \lambda^n$ $(f(n) \sim g(n) \Leftrightarrow \lim_{n\to\infty} f(n)/g(n) = 1)$
- $\lambda =$ dominating growth $\rightarrow$ find it!
- $s(n) =$ subexponential factor (we will ignore this one)

Generating functions

A generating function is a way of encoding an infinite sequence of numbers by treating them as the coefficients of a formal power series.

A generating function is a device somewhat similar to a bag.
Instead of carrying many little objects detachedly, which could be embarrassing,
we put them all in a bag, and then we have only one object to carry, the bag
(Pólya)

The rabbit counting a.k.a. Fibonacci numbers:

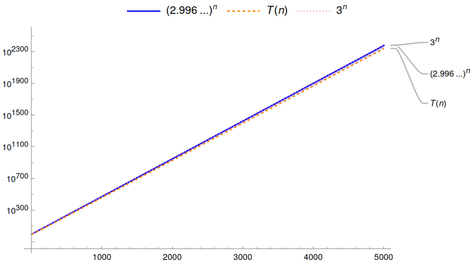
$$g(z) = \frac{1}{1-z-z^2} = \boxed{1}\,z^0 + \boxed{1}\,z^1 + \boxed{2}\,z^2 + \boxed{3}\,z^3 + \boxed{5}\,z^4 + \boxed{8}\,z^5 + \boxed{13}\,z^6 + \dots$$

- **Generating function** = function with Taylor expansion giving a fixed sequence
- **Example** The generating function for the square numbers n^2 is $g(z) = \frac{z(z+1)}{(1-z)^3}$
- **Fantastic fact** Radius of convergence $g(z)$ = dominating growth $^{-1}$ (unless the problem is crazy)

Enter, the theorem

The tree constant = dominating growth rate is

$$\lambda \approx 2.9955765856$$

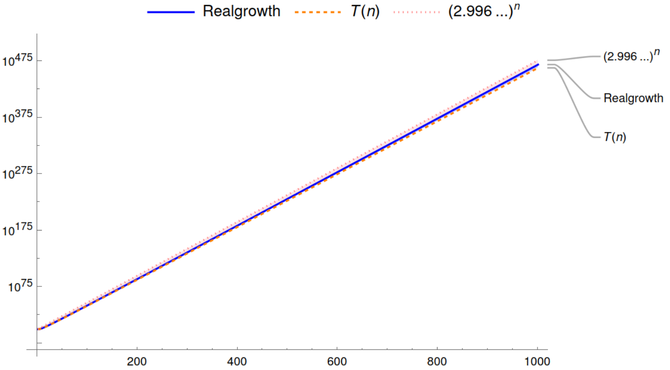


- This can be proven by studying the generating function $g(z)$:

$$\lambda^{-1} \text{ is the radius of convergence of } g(z)$$

- $g(z)$ is given by functional equation that is a bit annoying to write down, see A000055 on OEIS for details

The real growth rate



- We actually have

$$\boxed{T(n) \sim \beta \cdot n^{-5/2} \cdot \lambda^n}$$

- Here $\beta \approx 0.5349496061$ is a scalar computable from λ
- The point This can also be derived from the generating function (skipped)

Thank you for your attention!

I hope that was of some help.