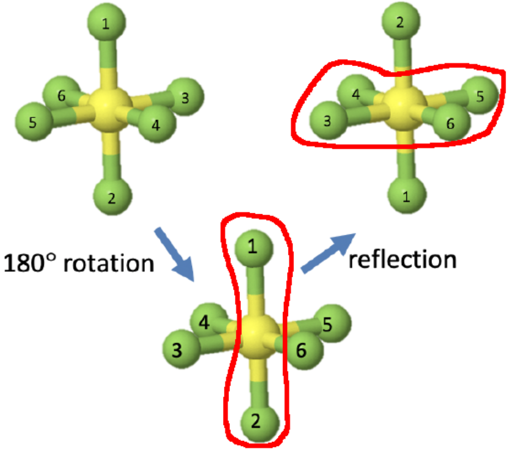
What is...Nagata's theorem?

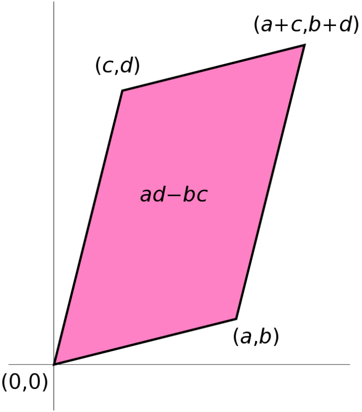
Or: Ill-behaved invariants

Invariant theory



- Invariant = something that remains unchanged
- Example The total energy of a system is invariant when the system evolves in time
- Invariant theory = the study of invariants of group symmetries on vector spaces

The determinant



- $SL_2(\mathbb{C})$ acts on 2-by-2 matrices by left multiplication
- The determinant *det* is an invariant under this action
- Theorem The ring of invariants is a polynomial ring in *det*

Hilbert's fourteenth problem

$$e_1(X_1, X_2, \ldots, X_n) = \sum_{1 \le j \le n} X_j,$$

$$e_2(X_1, X_2, \ldots, X_n) = \sum_{1 \le j < k \le n} X_j X_k,$$

$$e_3(X_1, X_2, \ldots, X_n) = \sum_{1 \le j < k < l \le n} X_j X_k X_l$$

- In 1900 Hilbert gave very influential 23 problems for the 20th century
- One of them is:

$G \subset GL_N(\mathbb{C})$ acts on $V = \mathbb{C}\{x_1, \ldots, x_n\}$, is $\mathbb{C}[x_1, \ldots, x_n]^G$ finitely generated?

- Example $G = S_2 \subset GL_N(\mathbb{C})$, then $\mathbb{C}[x_1, x_2]^G = \mathbb{C}[x_1 + x_2, x_1 x_2]$

Enter, the theorem

Hilbert's fourteenth problem is wrong

Let $x_1, \dots, x_{16}$, $t_1, \dots, t_{16}$ be algebraically independent elements over k and let G be the set of linear transformations σ such that (i) $\sigma(t_i) = t_i$ for any i and (ii) $\sigma(x_i) = x_i + b_i t_i$ with $(b_1, \dots, b_{16}) \in V^*$. Then:

The set $\mathfrak{o}$ of elements of $k[x_1, \dots, x_{16}, t_1, \dots, t_{16}]$ which are invariant under G is not finitely generated.

- Zariski showed ∼1956 that there are no counterexamples with ≤ 2 variables
- Nagata found the first counterexample in ∼1958 with ≥ 32 variables
- The minimal number of variables for which Hilbert's fourteenth problem is wrong depends on the precise question (there are several not quite equivalent formulations of Hilbert's fourteenth problem)

The setting is as follows: Assume that k is a field and let K be a subfield of the field of rational functions in n variables,

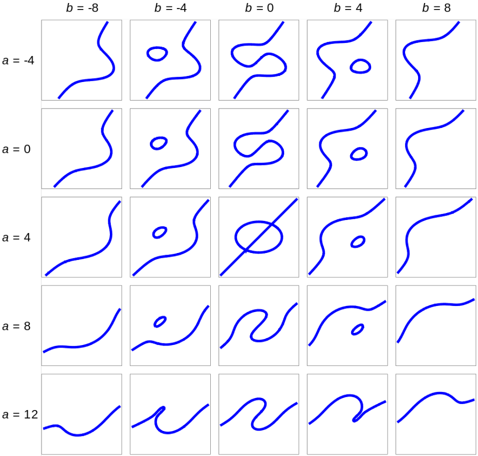
$k(x_1, \dots, x_n)$ over k.

Consider now the k-algebra R defined as the intersection

$$R := K \cap k[x_1, \dots, x_n] \, .$$

Hilbert conjectured that all such algebras are finitely generated over k.

Points on plane curves



- Lemma For any m there is no curve of degree $4m$ which goes through 16 generic points with multiplicity at least m
- This is the key to Nagata's counterexample
- Turns out that the method is more important than the counterexample itself

Thank you for your attention!

I hope that was of some help.