What is...Goodstein's theorem?

Or: Killing hydras

The Goodstein sequence

- $G_1(m) = m$ Initiation
- Write $G_n(m)$ hereditary base n and replace all n by $n+1$ Growth
- Subtract 1 Cut

$$35 = 2^{(2^{(2^{(1)})})} + 2^1 + 1 \xrightarrow{\text{grow}} 3^{3^{3^1}} + 3^1 + 1 \xrightarrow{\text{cut}} = 3^{3^{3^1}} + 3^1 = 7625597484990$$

$$7625597484990 = 3^{3^{3^1}} + 3^1 \xrightarrow{\text{grow}} 4^{4^{4^1}} + 4^1 \xrightarrow{\text{cut}} = 4^{4^{4^1}} + 4^1 - 1 = x \approx 10^{154}$$

$$x = 4^{4^{4^1}} + 3 \xrightarrow{\text{grow}} 5^{5^{5^1}} + 3 \xrightarrow{\text{cut}} 5^{5^{5^1}} + 2 \approx x = 10^{2185}$$

$$x = 5^{5^{5^1}} + 2 \xrightarrow{\text{grow}} 6^{6^{6^1}} + 2 \xrightarrow{\text{cut}} 6^{6^{6^1}} + 1 \approx x = 10^{36306}$$

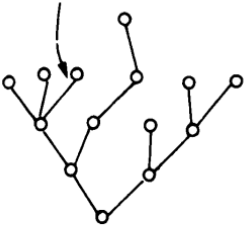
...

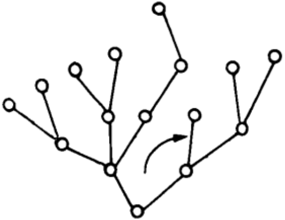
Question. Does this ever become zero?

Kirby–Paris hydra game — can you kill it?

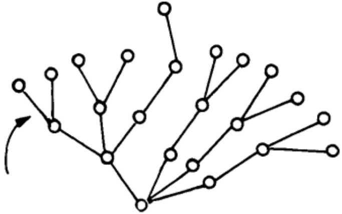
Rules

- A hydra is a rooted tree, and you can chop of one of its heads in any step
- If a non-rooted head is cut off in step n, the hydra grows n copies of the part above the branch node from which the head was removed

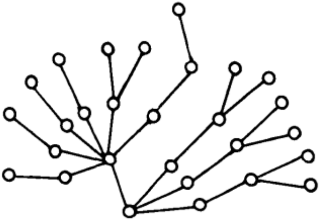




after stage 1

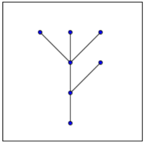
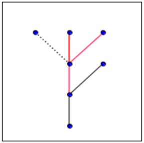
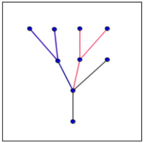
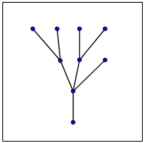
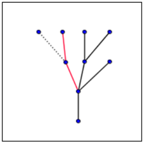
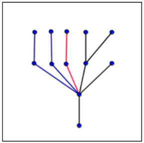
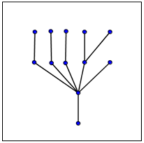
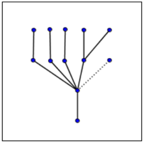
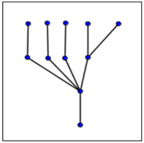
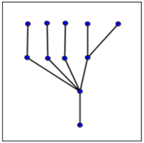
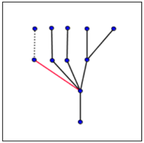
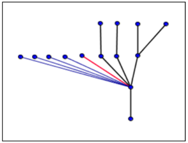


after stage 2



after stage 3

Let us kill the hydra

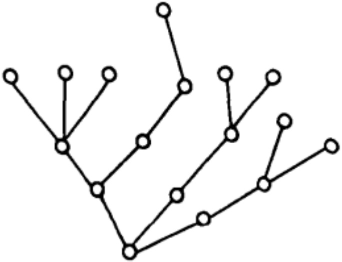
Step 1			
Step 2			
Step 3			
Step 4			

Enter, the theorem!

You will always kill the hydra, but you can not prove it:

- Every strategy is a winning strategy
- The statement "Every recursive strategy is a winning strategy" is not provable in classical arithmetic (Peano arithmetic)

You can prove it using stronger number system axioms!



$\omega^{\omega^{\omega}+\omega^{3}}+\omega^{\omega^{2}}\cdot 2$

after stage 1

$\omega^{\omega^{\omega}+\omega^{3}}+\omega^{\omega^{2}}+\omega^{\omega\cdot 3}$

after stage 2

When does Goodstein's sequence hits zero?

Start value	Hits zero at stage
1	2
2	4
3	6
4	$\approx 10^{10121210694}$
5	$\approx 10^{10^{10^{19728}}}$
$\vdots$	$\vdots$
12	$>$Graham's number
$\vdots$	$\vdots$

$$2 = 2 \xrightarrow{\text{grow}} 3 \xrightarrow{\text{cut}} = 3 - 1 = 2$$

$$2 = 2 \xrightarrow{\text{grow}} 2 \xrightarrow{\text{cut}} = 2 - 1 = 1$$

$$1 \xrightarrow{\text{grow}} 1 \xrightarrow{\text{cut}} 1 - 1 = 0$$

Thank you for your attention!

I hope that was of some help.