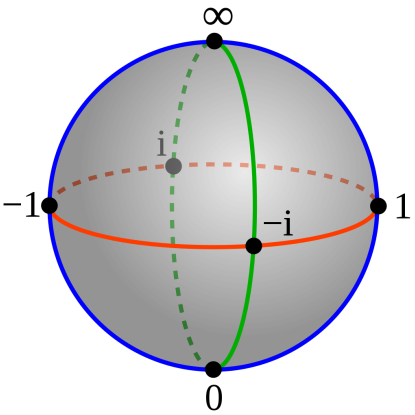
What is...the Hodge conjecture?

Or: Topology and algebraic geometry

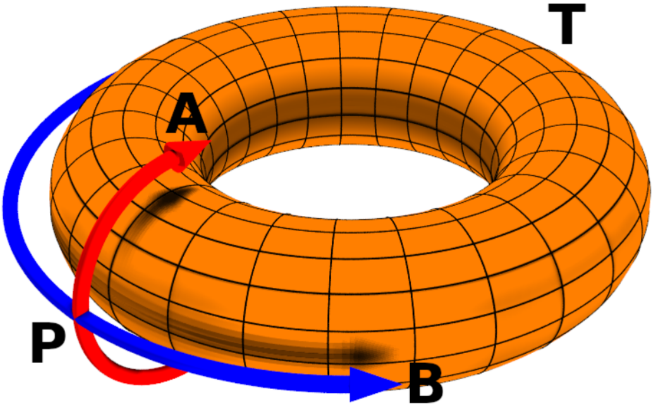
Topology meets geometry

The Riemann sphere is a complex projective manifold :



- Manifolds (=locally discs) are key objects in topology
- Projective varieties (=zero sets of homo. poly.) are key objects in geometry
- Question What happens if we put them together?
  We study complex projective manifolds

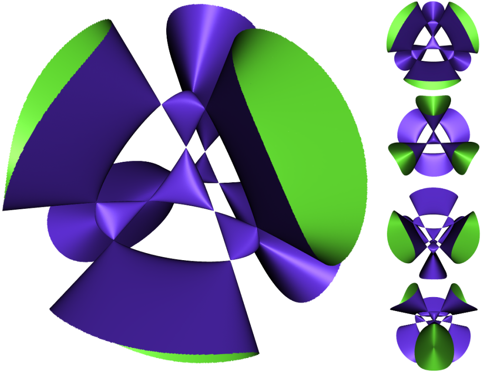
Topologist study homology



$$H_0(T) \cong \mathbb{Z} \leftrightsquigarrow [P], H_1(T) \cong \mathbb{Z}^2 \leftrightsquigarrow [A], [B], H_2(T) \cong \mathbb{Z} \leftrightsquigarrow [T]$$

- Let X be some manifold
- In good case (co)homology $H_*(X)$ or $H^*(X)$ counts submanifolds
- Example For the torus $H_*(T)$ finds the submanifolds $P = \textit{point}$, A, B and T

Algebraists study Chow groups



- Let X be some variety
- In good case the Chow group $CH^*(X)$ counts subvarieties
- Problem Chow groups notoriously difficult to compute: they are huge and mysterious
- Question When X is an manifold and a variety, is there any relation between submanifolds and subvarieties?

Enter, the theorem

Let X be a non-singular complex projective manifold of real dim $2n$ and:

(i) One can write $H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X)$ with $H^{p,q}(X) =$ (p,q)-forms

(ii) Define the Hodge classes:

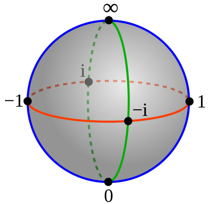
$$Hod^k = H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$$

(iii) Question Which Hodge classes are $\mathbb{Q}$-linear combinations of subvarieties?

(iv) Theorem All Hodge classes are for $k = 1$

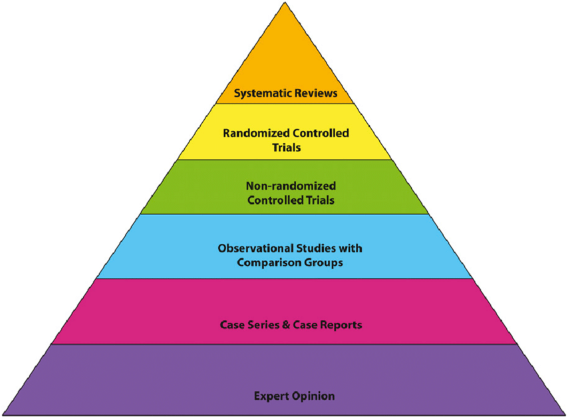
- Millenniums price problem Is the above theorem true for all k?
- For the Riemann sphere $H^*(S^2) \cong CH^*(S^2)$, so the conjecture is true

The Riemann sphere is a complex projective manifold :



A bit wonky...?

Where to place the Hodge conjecture?



- From the Millenniums price problem the Hodge conjecture is the one with the least evidence (in an imprecise sense)
- However there is evidence, just computations are very hard
- E.g. its true in dimension and codimension 1, but not much more is known

Thank you for your attention!

I hope that was of some help.