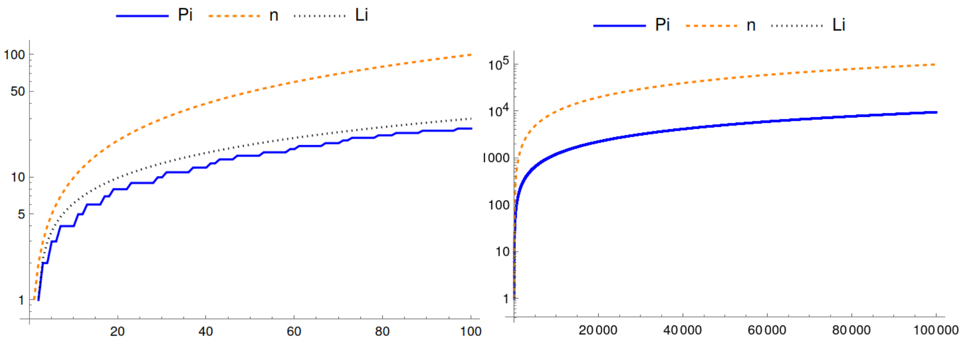
What is...the Riemann hypothesis?

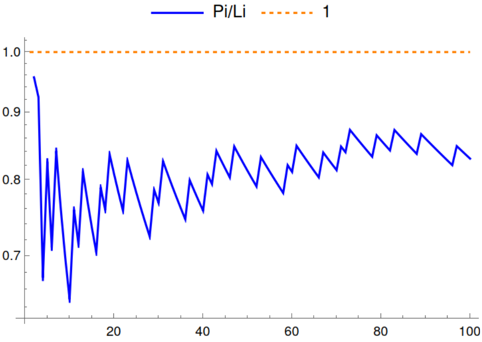
Or: Critical strip free (almost)

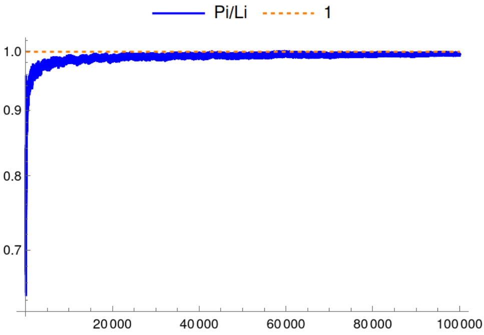
Counting primes - rough version



- Prime number function $\pi(n)$ = # primes $\leq$ n
- Counting primes precisely is very tricky as primes "pop up randomly"
- Question What is the leading growth of the number of primes?
- Answer There are roughly $c(n)n$ for sublinear correction term $c(n)$

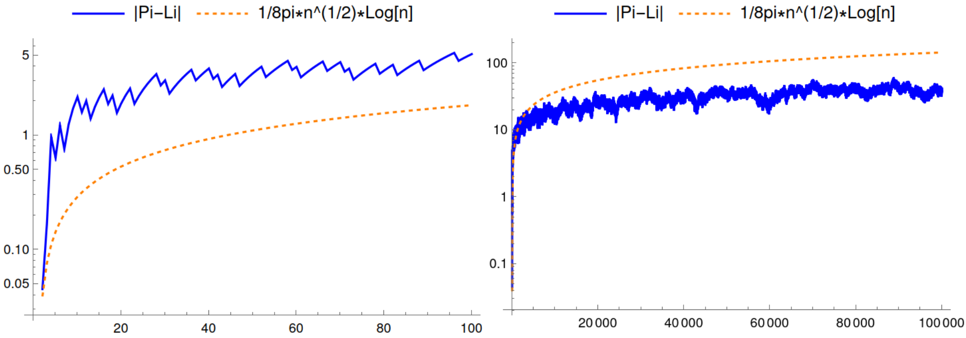
Counting primes - asymptotic version





- **Asymptotically equal** $f \sim g$ if $\lim_{n\to\infty} f(n)/g(n) \to 1$
- **Logarithmic integral** $Li(x) = \int_2^x 1/\ln(t)dt$
- **Question** What is the growth of the number of primes asymptotically?
- **Answer** We have $\pi(n) \sim n/\log(n) \sim Li(n)$

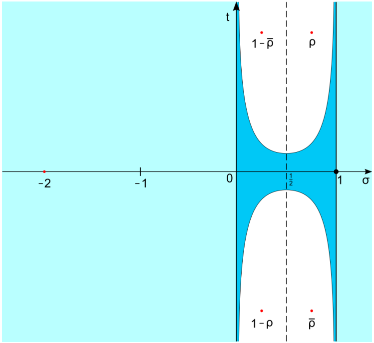
Counting primes - variance



- Asymptotically equal **does not** imply that the difference is good
- $|f(n) - g(n)|$ is a measurement of how goof the approximation is
- **Question** What is variance from the expected value $Li(n)$?
- **Conjectural answer** We have $|\pi(n) - Li(n)| \in O(n^{1/2} \log n)$ or $|\pi(n) - Li(n)| \leq \frac{1}{8\pi} n^{1/2} \log n$ (for $n \geq 2657$)

Enter, the theorem

Let $\beta = \sup\{Re(x) \mid x \text{ is a zero of } \zeta(s)\}$



Then $|\pi(n) - Li(n)| \in O(n^{\beta} \log n)$

- $\zeta \colon \mathbb{C} \setminus \{1\} \to \mathbb{C}$ is the Riemann zeta function and it is the meromorphic continuation of $s \mapsto \sum_{n=1}^{\infty} n^{-s}$
- The Riemann hypothesis conjectures that $\beta = 0.5$

Not quite the Riemann hypothesis

VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Durch Einsetzung dieser Werthe in den Ausdruck von $f(x)$ erhält man

$$f(x) = Li\,(x) - \Sigma^{\alpha}\left(Li\,(x^{\frac{1}{2}+\alpha i}) + Li\,(x^{\frac{1}{2}-\alpha i})\right)$$

$$+ \int_x^{\infty} \frac{1}{x^2-1}\,\frac{dx}{x \log x} + \log \xi\,(0),$$

wenn in Σ^{α} für α sämmtliche positiven (oder einen positiven reellen Theil enthaltenden) Wurzeln der Gleichung $\xi\,(\alpha) = 0$, ihrer Grösse nach geordnet, gesetzt werden. Es lässt sich, mit Hülfe einer genaueren Discussion der Function ξ, leicht zeigen, dass bei dieser Anordnung der Werth der Reihe

- A bit more work shows $|\pi(n) - Li(n)| \leq \frac{1}{8\pi} n^{1/2} \log n$ (for $n \geq 2657$)
- Calling this the Riemann hypothesis is far fetched but I go for it anyway (sorry): its in the spirit of Riemann's paper and somewhat easier to understand

Thank you for your attention!

I hope that was of some help.