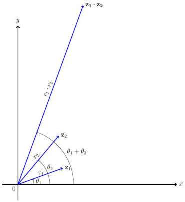
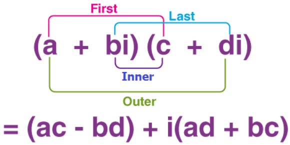
What are...octonions?

Or: Division = good, associativity = bad

Double once ⇒ all good





- **Doubling** = Take an algebra A with (anti)involution * and create a new algebra with involution $B = A \oplus A$ via

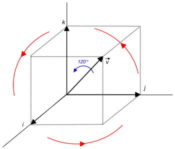
$$(p, q)(r, s) = (pr - s^* q, sp + qr^*), \quad (p, q)^* = (p^*, -q)$$

- **Example** For $A = \mathbb{R}$ with $^* = id$ doubling gives $B = \mathbb{C}$
- **Complex numbers** $\mathbb{C}$ ⇝ 2d number system

Double twice ⇒ not commutative

Quaternion multiplication table

	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1



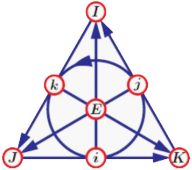
- Doubling = Take an algebra A with (anti)involution * and create a new algebra with involution $B = A \oplus A$ via

$$(p,q)(r,s) = (pr - s^*q, sp + qr^*), \quad (p,q)^* = (p^*, -q)$$

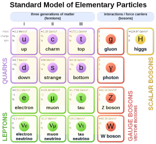
- Example For $A = \mathbb{C}$ with *=complex conjugation doubling gives $B = \mathbb{H}$

- Quaternions $\mathbb{H}$ $\rightsquigarrow$ noncommutative 4d number system

Triple once ⇒ not commutative + not associative



×	**i**	**j**	**k**	**E**	**I**	**J**	**K**
i	−1	k	−j	I	−E	−K	J
j	−k	−1	i	J	K	−E	−I
k	j	−i	−1	K	−J	I	−E
E	−I	−J	−K	−1	i	j	k
I	E	−K	J	−i	−1	−k	j
J	K	E	−I	−j	k	−1	−i
K	−J	I	E	−k	−j	i	−1



- **Doubling** = Take an algebra A with (anti)involution * and create a new algebra with involution $B = A \oplus A$ via

$$(p,q)(r,s) = (pr - s^*q, sp + qr^*), \quad (p,q)^* = (p^*, -q)$$

- **Example** For $A = \mathbb{H}$ with *=quaternion conjugation doubling gives $B = \mathbb{O}$
- **Octonions** $\mathbb{O}$ $\rightsquigarrow$ noncommutative + nonassociative 8d number system

Enter, the theorem

We have the following:

(i) $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are normed division algebras over $\mathbb{R}$ Invertibility

(ii) Their dimensions are $1, 2, 4, 8$ This sequence appears somehow everywhere

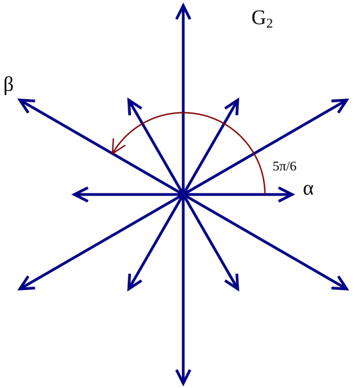
(iii) There are no other normed division algebras over $\mathbb{R}$ $\mathbb{O}$ is maximal

Any further doubling process looses the invertibility

- Normed division algebra = every nonzero element is invertible, there is a norm
- Only a few properties survive doubling in general, e.g. power associativity does
- Here is the 4th doubling $\mathbb{S}$:

e_ie_j		e_j															
		e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
	e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
	e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
	e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
	e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
	e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
	e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
	e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
	e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_i	e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	$-e_0$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	$-e_0$	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
	e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	$-e_0$	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
	e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
	e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
	e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	$-e_0$	e_3	$-e_2$
	e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	$-e_0$	e_1
	e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	$-e_0$

Octonions everywhere...!?



- $\mathbb{O}$ magically appears in the classification and construction of many exceptional mathematical objects
- Example The Lie group G_2 is obtained from automorphisms of $\mathbb{O}$
- Example $\mathbb{O}$ provides an elementary derivation of the Leech lattice

Thank you for your attention!

I hope that was of some help.