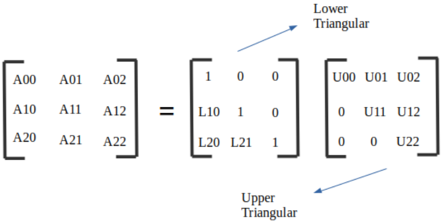
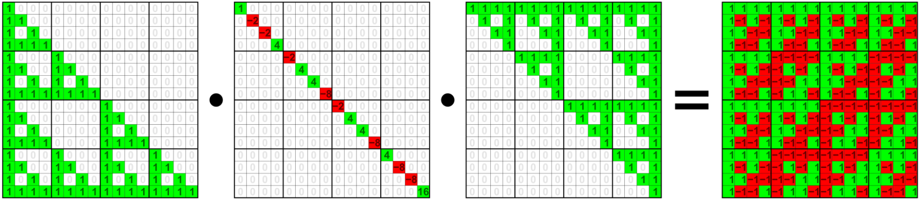
What is...the Bruhat decomposition?

Or: Lower and upper

The lower-upper (LU) decomposition

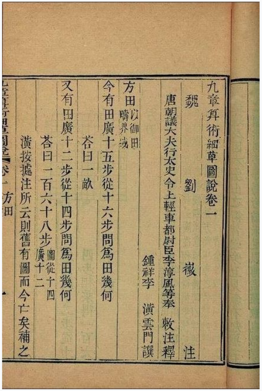




- LU decomposition = we can write a matrix as a product $M = LU$
- L is lower triangular lower triangular; U is upper triangular
- In general we need a permutation matrix P as well and $M = PLU$

Gaussian elimination

The Nine Chapters on the Mathematical Art



A page of *The Nine Chapters on the Mathematical Art* (1820 edition)

- That the LU decomposition works follows from Gaussian elimination
- The LU decomposition origins are hence early on e.g. in "The Nine Chapters on the Mathematical Art" ∼10th-2nd century BCE

Turning lower into upper

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & b \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$L = P(w_0)UP(w_0)$$

- Observation We can always turn a lower matrix into an upper one
- The price we pay doing this are permutation matrices for the longest permutation w_0

Enter, the theorem

We have the Bruhat decomposition:

$$G = \bigcup_{w \in W} BwB$$

where $G = \mathrm{GL}_n(\overline{\mathbb{K}})$ = invertible n-by-n matrices, B = upper triangular matrices, W = symmetric group in $\{1, \ldots, n\}$

- Example

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 4/3 \\ 0 & 2/3 \end{pmatrix}$$

Actually its much more general

Type	Lie algebra
A_n	$\mathfrak{sl}_{n+1}$
B_n	$\mathfrak{so}_{2n+1}$
C_n	$\mathfrak{sp}_{2n}$
D_n	$\mathfrak{so}_{2n}$

$\mathfrak{g}$	W	$\lvert W\rvert$
A_r	S_{r+1}	$(r+1)!$
B_r	$\mathbb{Z}_2^r \rtimes \mathrm{S}_r$	$2^r\, r!$
C_r	$\mathbb{Z}_2^r \rtimes \mathrm{S}_r$	$2^r\, r!$
D_r	$\mathbb{Z}_2^{r-1} \rtimes \mathrm{S}_r$	$2^{r-1}\, r!$

- The Bruhat decomposition works actually very general
- G = connected, reductive algebraic group over an algebraically closed field; B = Borel; W = Weyl group
- In this case we still have $G = \bigcup_{w\in W} BwB$

Thank you for your attention!

I hope that was of some help.