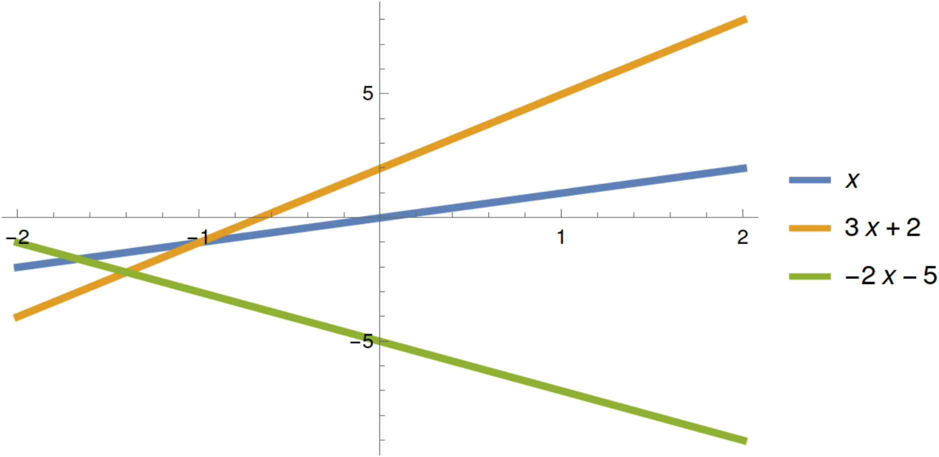
What is...the Ax–Grothendieck theorem?

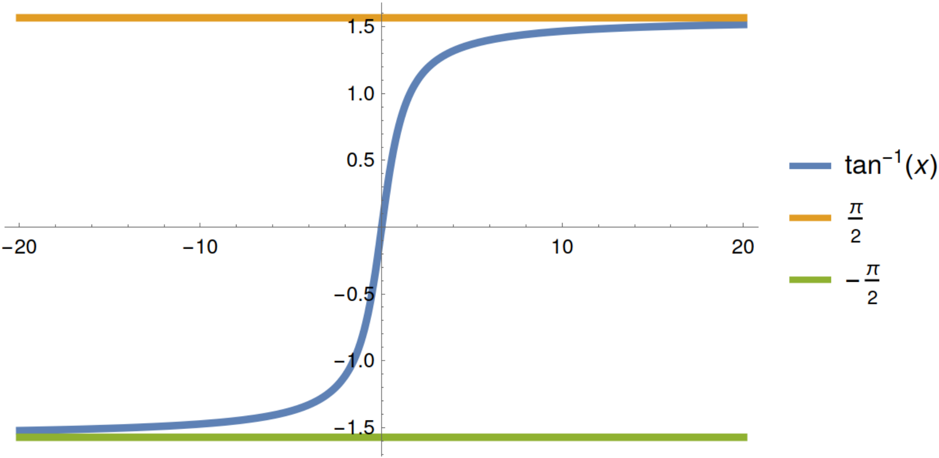
Or: Polynomials rule!

The linear setting



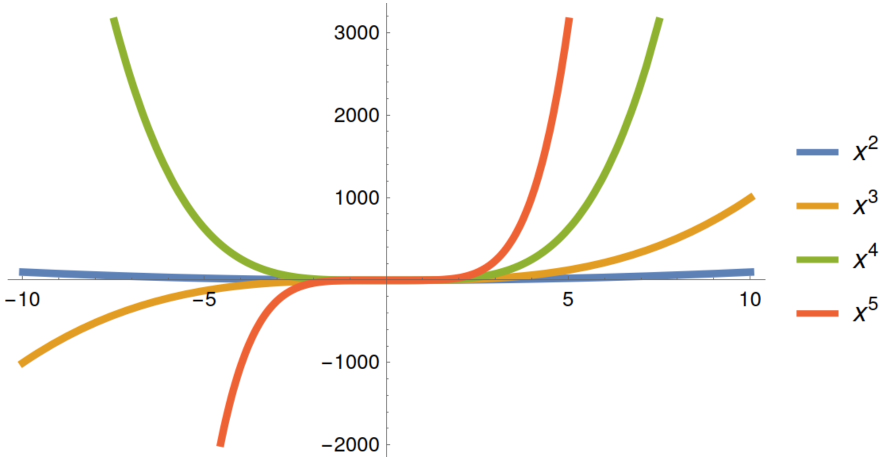
- (Affine) linear functions hit any real number unless they have slope zero
- Slope zero = not injective
- Hence, injective ⇒ bijective

The general setting



- General (smooth) functions do not satisfy injective ⇒ bijective
- For example, arctan is injective but not bijective
- This cannot be fixed by going to the complex numbers (actually the complex arctan is multi-valued)

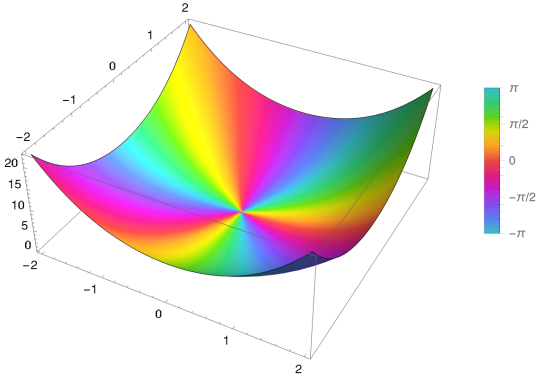
The polynomial setting



- General polynomial functions do satisfy injective ⇒ bijective
- For example, x^{odd} is injective and bijective
- This stays the same for $\mathbb{C} \to \mathbb{C}$ polynomials

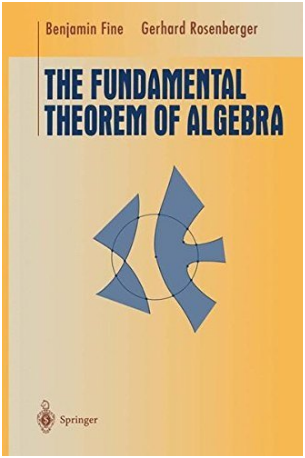
Enter, the theorem

Any injective polynomial function $\mathbb{C}^n \to \mathbb{C}^n$ is bijective



- The same is true for $\mathbb{R}^n \to \mathbb{R}^n$ but the proof is more difficult
- The full theorem generalizes to any algebraic variety over an algebraically closed field, e.g. for $\overline{\mathbb{F}_p}$

This extends the fundamental theorem of algebra (FToA)



- Proof (sketch) for $n = 1$ Fix an injective polynomial p
- Injectivity implies that, for all $z_0 \in \mathbb{C}$, the function $p(z) - z_0$ is not constant
- FToA $\Rightarrow p(z) - z_0 = 0$ for some $z \in \mathbb{C}$, so $p(z) = z_0$ for some $z \in \mathbb{C}$

Thank you for your attention!

I hope that was of some help.