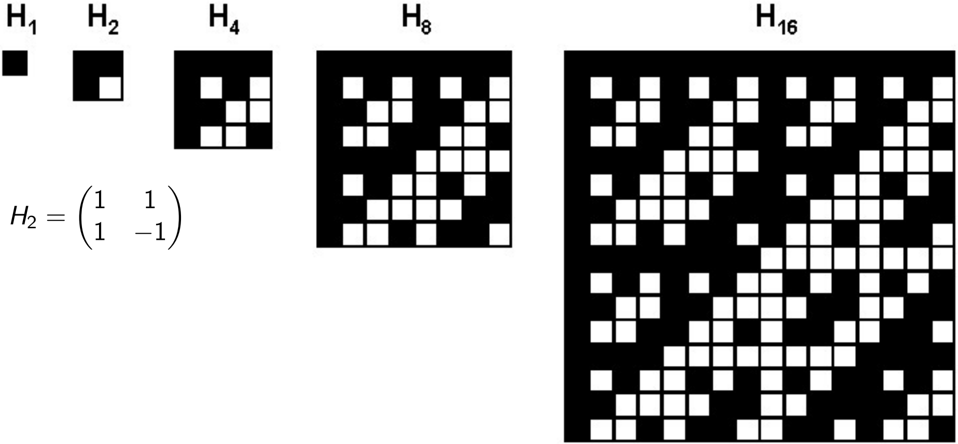
What is...a Hadamard matrix?

Or: Orthogonality exists! Or not?

The setting

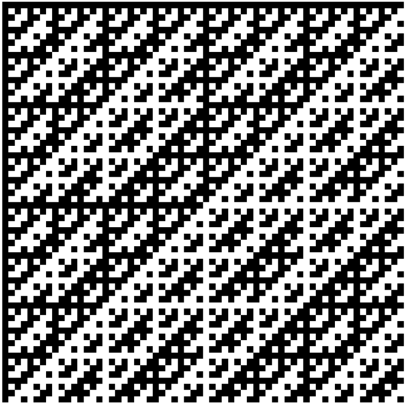


$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- **Hadamard matrix** = only entries $+1$ and -1, rows are mutually orthogonal
- **Illustration** $+1$ and -1 get two different colors
- **Question** Do these matrices exist?

First constructions

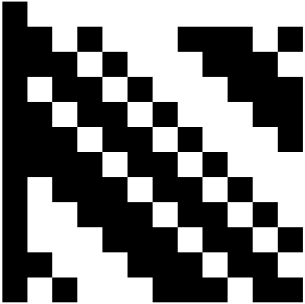
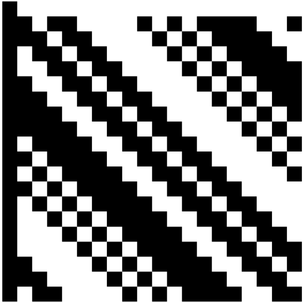
$n = 64$:



- **Easy fact** Hadamard matrices can only exist for $n = 1, 2$ or multiples of 4
- **For $n = 2^k$** Use the construction

$$H_{2^{k+1}} = \begin{pmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{pmatrix}$$

$n \neq 2^k$ gets rather random

$n = 12$:  , $n = 20$: 

$n = 84$: 

- For $n = 12, 20$ there are Hadamard matrices that look kind of random
- It gets even worse for bigger n
- Nontrivial problem Do Hadamard matrices of order $4k$ always exist?

Enter, the theorem

Hadamard matrices exist for:

- ▷ $n = 2^k$ Easy
- ▷ $q + 1$ for q a prime power $\equiv 3 \bmod 4$ or $2(q + 1)$ for q a prime power $\equiv 1 \bmod 4$ Not too bad
- ▷ Some sporadic values such as 92 Quite hard

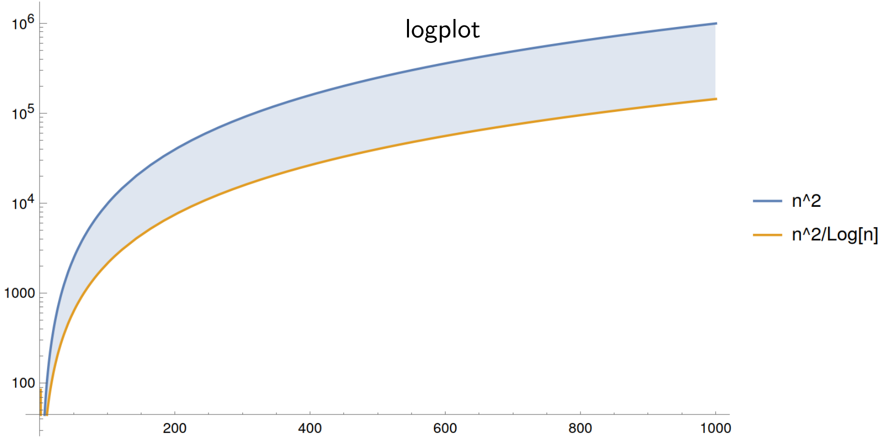
$n = 92$: 

- ▶ The Hadamard conjecture

  Do Hadamard matrices of order $4k$ always exist?

  is a major unsolved problem in mathematics
- ▶ The smallest open value is $n = 668$ (in early 2023)

Very overdetermined



- The entries of an Hadamard matrix are tightly coupled; there is a lot of redundant information
- If an Hadamard matrix of order n has $O(n^2/\log n)$ entries randomly deleted, then with overwhelming likelihood, one can recover the original matrix

Thank you for your attention!

I hope that was of some help.