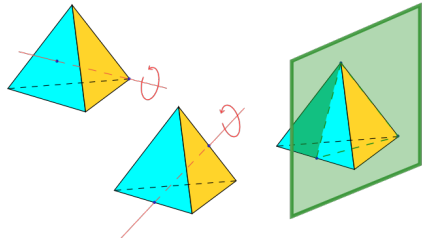
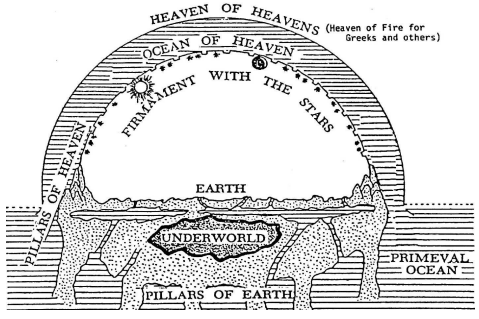
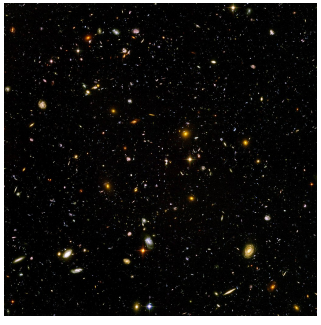
What are...independence results?

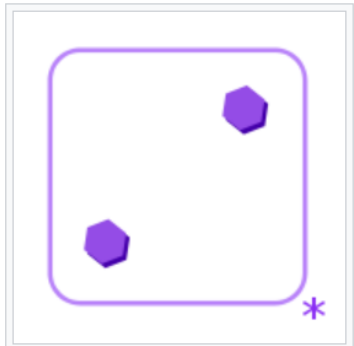
Or: Its true and its not true

Axioms and models

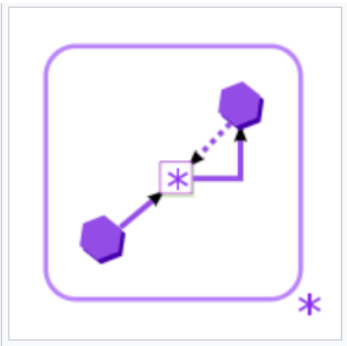
	Axioms	Model
Numbers	Peano axioms	$\mathbb{N}$ or  or...
Groups	Group axioms	$S_4 = \langle s, t, u \mid \text{some relations} \rangle$ or...  or...
Sets	ZFC axioms	 or  or...
More	...	...

- A model is a realization of some axiom system
- Example Any group is a model of the group axioms

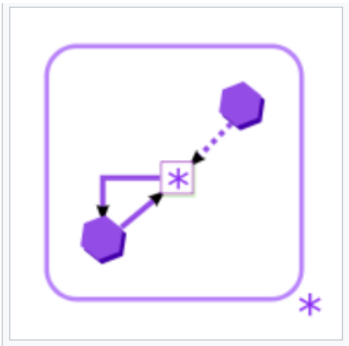
"Truth"



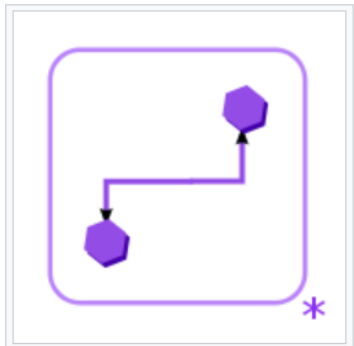
1. Assume a group has two identities.



2. $e_1 * e_2 = e_1$
as e_2 is identity of G,
and e_1 is in G.



3. $e_1 * e_2 = e_2$
as e_1 is identity of G,
and e_2 is in G



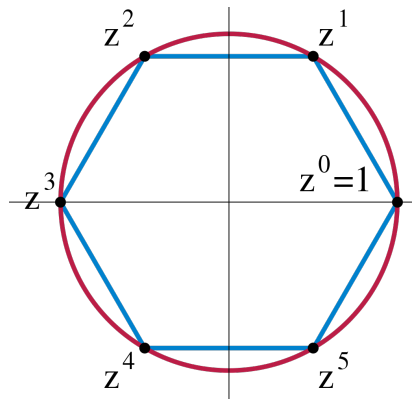
4. The two identities are the same.

5. a group only has one identity

- There are statements that follow from the axioms
- Example "Uniqueness of the unit"' follows from the group axioms
- Such statements are true in all models

Independence

The existence of $\mathbb{Z}$ $\Rightarrow$ Y does not follow from GA

The existence of 

$\Rightarrow$ $\neg Y$ does not follow from GA

- A statement X is independent of an axiom system A if

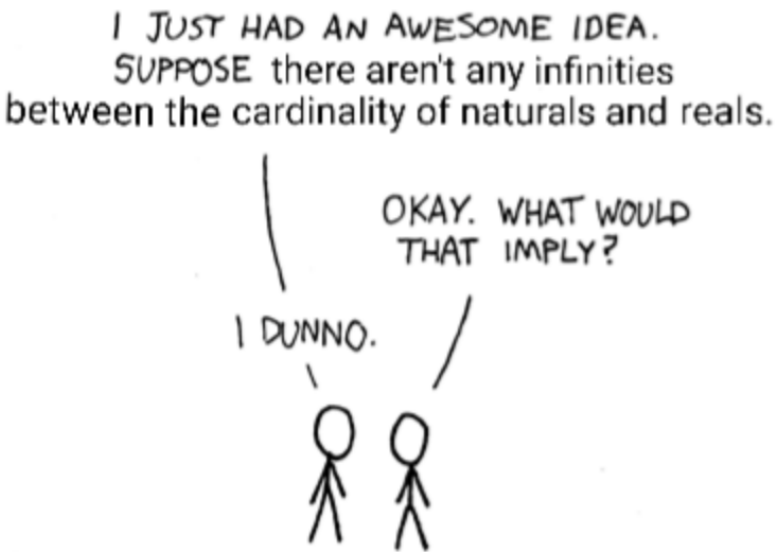
  neither X nor $\neg X$

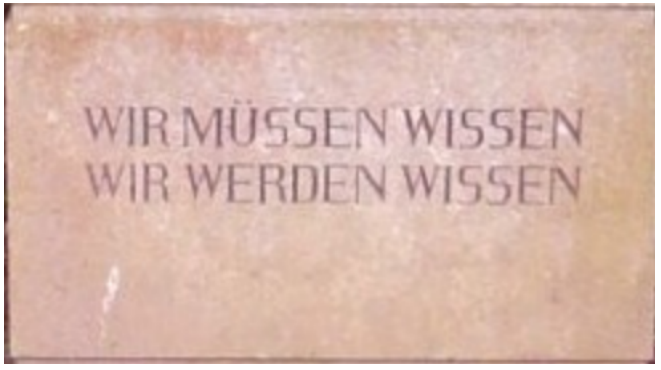
  can be proven in A

- Example $Y =$ "$\exists$ an element $\neq 1$ of finite order" is independent of the group axioms GA

Enter, the theorem

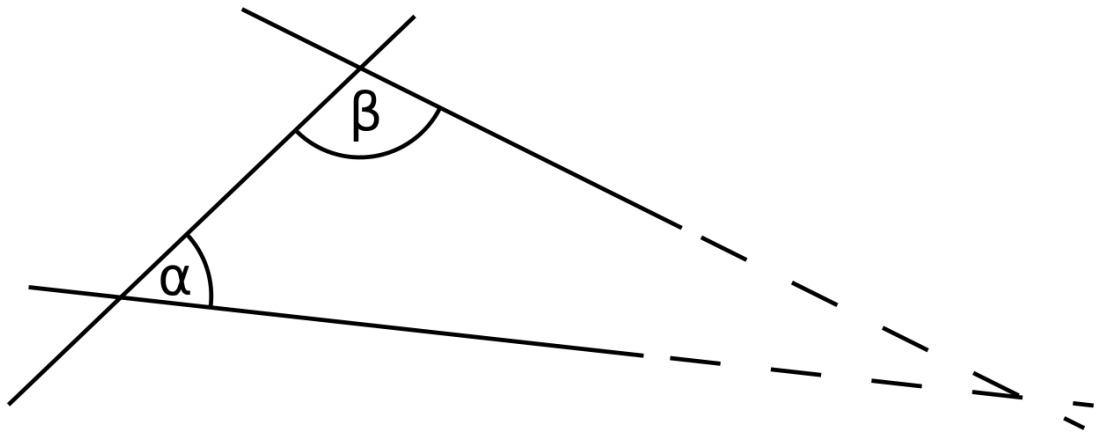
The CH is independent of ZFC





- This is maybe the most important independence result and #1 of Hilbert's problems
- CH = continuum hypothesis = there is no set whose cardinality is strictly between that of $\mathbb{N}$ and $\mathbb{R}$
- ZFC = Zermelo–Fraenkel plus choice = standard axioms of set theory
- Proof Gödel ∼1940: CH holds in some model; Cohen ∼1963: ¬CH holds in some model

More independence



- There are many more independence results
- Example The parallel postulate PP is independent of the other axioms of Euclid
- Proof Folklore ∼xxBC: PP holds in some model; Gauss+friends ∼1800: ¬PP holds in some model

Thank you for your attention!

I hope that was of some help.