

What is...the discrete periodic table?

Or: Finite simple groups

The discrete periodic table – the finite simple groups

$0, C_n, Z_n$ 1 1	Dynkin Diagrams of Simple Lie Algebras										C_2 2						
$A_1(4), A_1(5)$ A_5 60	$A_2(2)$ $A_1(7)$ 168											C_3 3					
$A_4(9), B_2(2)'$ A_6 360	${}^2C_3(3)'$ $A_1(8)$ 504	$B_2(3)$ 25920	$C_3(3)$ 1485 351 688	$D_4(2)$ 174 182 400	${}^2D_4(2^2)$ 197 406 720	$G_2(2)'$ ${}^2A_2(9)$ 6 018	C_5 5										
A_7 2520	$A_1(11)$ 660	$E_6(2)$ 214 841 375 922 605 275 220 480	$E_7(2)$ 1 041 044 63 746 108 667 243 826 671 840	$E_8(2)$ 3 113 126 63 746 108 667 243 826 671 840	$F_4(2)$ 3 113 126 63 746 108 667 243 826 671 840	$G_2(3)$ 4 245 696	${}^3D_4(2^3)$ 211 341 312	${}^2E_6(2^2)$ 76 532 479 605	${}^2B_2(2^3)$ 29 120	${}^2F_4(2)'$ 17 971 200	${}^2G_2(3^3)$ 10 273 444 472	$B_3(2)$ 1 431 520	$C_4(3)$ 63 746 108 654 489 600	$D_5(2)$ 23 499 295 948 800	${}^2D_5(2^2)$ 25 015 379 508 400	${}^2A_2(25)$ 126 000	C_7 7
$A_8(2)$ A_8 20 160	$A_1(13)$ 1 092	$E_6(3)$ 1 107 107 367 163 483 522 605 275 220 480 000	$E_7(3)$ 1 374 470 608 000 000 67 532 479 605 000 243 826 671 840 000	$E_8(3)$ 3 734 420 782 816 671 944 781 600	$F_4(3)$ 3 734 420 782 816 671 944 781 600	$G_2(4)$ 251 596 800	${}^3D_4(3^3)$ 20 560 831 748 912	${}^2E_6(3^2)$ 76 532 479 605 915 308 121 171 600	${}^2B_2(2^5)$ 32 537 600	${}^2F_4(2^3)$ 264 965 352 699 656 175 614 400	${}^2G_2(3^5)$ 49 825 657 639 348 552	$B_2(5)$ 4 680 000	$C_3(7)$ 273 457 218 681 953 400	$D_4(5)$ 8 911 539 000 195 648 800	${}^2D_4(4^2)$ 67 536 471	${}^2A_3(9)$ 3 285 920	C_{11} 11
A_9 151 440	$A_1(17)$ 2 448	$E_6(4)$ 65 025 107 164 000 605 275 220 480 000 243 826 671 840 000	$E_7(4)$ 1 041 044 63 746 108 667 243 826 671 840 000	$E_8(4)$ 3 113 126 63 746 108 667 243 826 671 840 000	$F_4(4)$ 3 113 126 63 746 108 667 243 826 671 840 000	$G_2(5)$ 5 899 000 000	${}^3D_4(4^3)$ 47 802 350 642 798 600	${}^2E_6(4^2)$ 46 896 121 716 49 825 657 642 798 600	${}^2B_2(2^7)$ 34 093 383 600	${}^2F_4(2^5)$ 1 041 044 63 746 108 667 243 826 671 840 000	${}^2G_2(3^7)$ 299 109 910 264 332 348 932 652	$B_2(7)$ 118 297 600	$C_3(9)$ 54 025 571 002 605 275 220 480	$D_5(3)$ 1 289 512 700 941 365 139 200	${}^2D_4(5^2)$ 17 880 201 256 600 000 000	${}^2A_2(64)$ 5 515 776	C_{13} 13
A_n at 2	$A_n(q)$ $\frac{q^n-1}{q-1}$	$E_6(q)$ $\frac{q^6-1}{q-1} \prod_{i=2}^6 (q^2-1)$	$E_7(q)$ $\frac{q^7-1}{q-1} \prod_{i=2}^7 (q^2-1)$	$E_8(q)$ $\frac{q^8-1}{q-1} \prod_{i=2}^8 (q^2-1)$	$F_4(q)$ $\frac{q^4-1}{q-1} \prod_{i=2}^4 (q^2-1)$	$G_2(q)$ $q^2(q^2-1)$	${}^3D_4(q^3)$ $\frac{q^4-1}{q-1} \prod_{i=2}^4 (q^2-1)$	${}^2E_6(q^2)$ $\frac{q^6-1}{q-1} \prod_{i=2}^6 (q^2-1)$	${}^2B_2(2^{2n+1})$ $q^{2n+1}(q-1)$	${}^2F_4(2^{2n+1})$ $\frac{q^{2n+1}-1}{(q^2-1)(q-1)}$	${}^2G_2(3^{2n+1})$ $q^{2n+1}(q-1)$	$B_n(q)$ $\frac{q^{2n}-1}{q-1} \prod_{i=2}^n (q^2-1)$	$C_n(q)$ $\frac{q^{2n}-1}{q-1} \prod_{i=2}^n (q^2-1)$	$D_n(q)$ $\frac{q^{2n}-1}{q-1} \prod_{i=2}^n (q^2-1)$	${}^2D_n(q^2)$ $\frac{q^{2n}-1}{q-1} \prod_{i=2}^n (q^2-1)$	${}^2A_n(q^2)$ $\frac{q^{2n}-1}{q-1} \prod_{i=2}^n (q^2-1)$	Z_p C_p p

■ Alternating Groups
■ Classical Chevalley Groups
■ Chevalley Groups
■ Classical Steinberg Groups
■ Steinberg Groups
■ Suzuki Groups
■ Ree Groups and Tits Groups*
■ Sporadic Groups
■ Cyclic Groups

Alternates*	Symbol	Order†
M_{11}	M_{12}	M_{22}
M_{23}	M_{24}	$J(1), J(11)$
J_1	J_2	HJ
J_3	J_4	HJM
HS	McL	F_4, BHH, BTH
He	Ru	

*This group ${}^2G_2'$ is not a group of Lie type, but is the Suzuki 2-constrained subgroup of ${}^2G_2(3)$. It is usually given honorary Lie type status.
 †For sporadic groups and families, alternate names in the upper left are other names by which they may be known. For specific non-sporadic groups these are used to indicate nonfamilies. All such nonfamilies appear on the table except the families $B_n(2^m) \cong C_n(2^m)$.

The groups starting on the second row are the classical groups. The sporadic maximal group is unrelated to the families of Suzuki groups.
 †The simple groups are denoted by their order with the following exceptions:
 $B_n(2)$ and $C_n(2)$ for $n > 2$,
 $A_n \cong A_2(2)$ and $A_1(4)$ of order 2004.

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Sz	$O'NS, O-S$	-3	-2	-1	F_4, D	L_3S	Ly	F_4, E	Th	$M(22)$	$M(23)$	$F_{4,3}, M(24)'$	F_2	B	F_4, M
Sz_2	$O'N$	C_{03}	C_{02}	C_{01}	HN	HN	Ly	Th	F_{i22}	F_{i23}	F_{i24}'	B	B	M	
840 545 497 900	666 915 305 430	695 746 656 000	42 345 621 312 000	4 157 776 806 543 364 000	273 050 912 900 000	51 745 379 004 000 000	90 745 943 987 072 000	61 561 754 654 000	269 800 000	4 089 478 475 1 255 205 789 300	661 721 262 800	1 170 740 401 226 000 100 000 000 000 000	100 000 000 000 000 100 000 000 000 000		

This awesome illustration by Ivan Adrias condenses a complicated statement - lets have a closer look!

Elements and (finite) simple groups

Chemistry	Group theory
Compounds	Groups
Elements	Simple groups
Simpler substances	Jordan–Hölder theorem
Periodic table	Classification of simple groups

► Simple groups are groups without normal subgroups **No substructure**

► For every group G there exists a (up to renaming) unique sequence

$$1 = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_n = G, \quad H_{i+1}/H_i \text{ simple} \quad \text{Building blocks}$$

► Example. For abelian groups the simples are “prime factors”, e.g.

$$1 \triangleleft \mathbb{Z}/2\mathbb{Z} \triangleleft \mathbb{Z}/6\mathbb{Z}, \quad 1 \triangleleft \mathbb{Z}/3\mathbb{Z} \triangleleft \mathbb{Z}/6\mathbb{Z}, \quad \text{Simples: } \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$$

► Example. Symmetric groups are almost simple, but they have a **center**:

$$1 \triangleleft \mathbb{Z}/2\mathbb{Z} \triangleleft S_n, \quad 1 \triangleleft A_n \triangleleft S_n, \quad \text{Simples } (n \geq 5): \mathbb{Z}/2\mathbb{Z}, A_n$$

The natural candidates are almost good

- ▶ Take $SL_n(\mathbb{C})$, which is basically a simple Lie group
- ▶ Replace \mathbb{C} by \mathbb{F}_q (field with q elements), get $SL_n(\mathbb{F}_q)$
- ▶ $SL_n(\mathbb{F}_q)$ is a finite group with $q(q^2 - 1)$ elements
- ▶ $SL_n(\mathbb{F}_q)$ is almost a simple finite group, *i.e.* for q odd

$$1 \triangleleft \mathbb{Z}/2\mathbb{Z} \triangleleft SL_n(\mathbb{F}_q)$$

with $\mathbb{Z}/2\mathbb{Z} = \{id, -id\} = \text{center}$. The quotient

$$SL_n(\mathbb{F}_q)/\text{center}$$

is simple unless $q = 2, 3$

- ▶ Something similar works for all simple Lie groups, *e.g.* $SO_n(\mathbb{F}_q)$ or $SP_{2n}(\mathbb{F}_q)$, by work of Chevalley and Steinberg

Enter, the theorem! (A short version of it.)

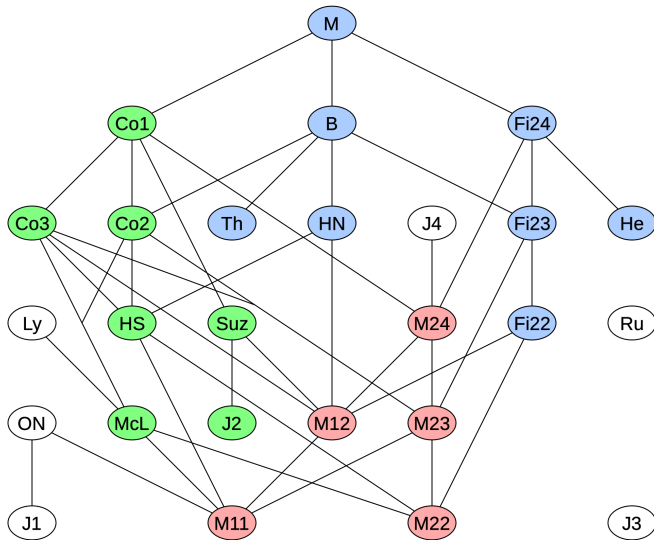
The discrete periodic table is complete:

- (a) The cyclic groups $\mathbb{Z}/p\mathbb{Z}$ are simple for p prime (infinitely many)
 - (b) The alternating groups A_n are simple for $n \geq 5$ (infinitely many)
 - (c) There are 16 families of finite simple groups of Lie type (infinitely many)
 - (d) There are 26 (or 27) exceptional finite simple groups (finitely many)
 - (e) There are no other simples This is the real meat
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Note that almost all simple groups can be constructed using the smooth periodic table of simple Lie groups/Lie algebras

In some sense even the A_n are of Lie type, using that " S_n is $GL_n(\mathbb{F}_1)$ "

Exotic discrete symmetries



The 26 sporadic simple groups correspond to “exotic symmetries” which have their vary unique place in nature

Thank you for your attention!

I hope that was of some help.