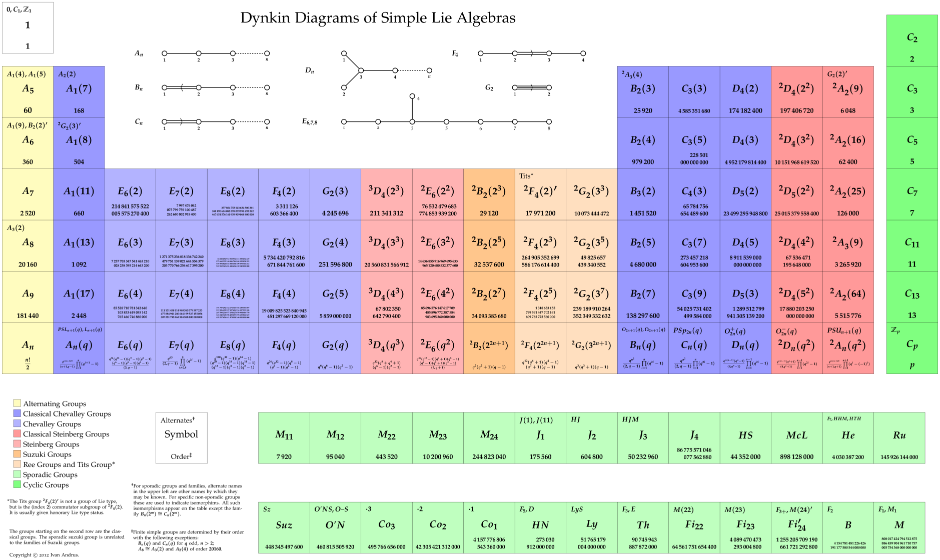
**What is...the discrete periodic table?**

Or: Finite simple groups

# The discrete periodic table – the finite simple groups




This awesome illustration by Ivan Andrus condenses a complicated statement - lets have a closer look!

## Elements and (finite) simple groups

| Chemistry | Group theory |
|---|---|
| Compounds | Groups |
| Elements | Simple groups |
| Simpler substances | Jordan–Hölder theorem |
| Periodic table | Classification of simple groups |

- Simple groups are groups without normal subgroups No substructure
- For every group $G$ there exists a (up to renaming) unique sequence

$$1 = H_0 \lhd H_1 \lhd \dots \lhd H_n = G, \quad H_{i+1}/H_i \text{ simple}$$ Building blocks

- Example. For abelian groups the simples are "prime factors", *e.g.*

$$1 \lhd \mathbb{Z}/2\mathbb{Z} \lhd \mathbb{Z}/6\mathbb{Z}, \quad 1 \lhd \mathbb{Z}/3\mathbb{Z} \lhd \mathbb{Z}/6\mathbb{Z}, \quad \text{Simples: } \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}$$

- Example. Symmetric groups are almost simple, but they have a center:

$$1 \lhd \mathbb{Z}/2\mathbb{Z} \lhd S_n, \quad 1 \lhd A_n \lhd S_n, \quad \text{Simples } (n \geq 5)\text{: } \mathbb{Z}/2\mathbb{Z}, A_n$$

## The natural candidates are almost good

- Take $\mathrm{SL}_n(\mathbb{C})$, which is basically a simple Lie group
- Replace $\mathbb{C}$ by $\mathbb{F}_q$ (field with $q$ elements), get $\mathrm{SL}_n(\mathbb{F}_q)$
- $\mathrm{SL}_n(\mathbb{F}_q)$ is a finite group with $q(q^2-1)$ elements
- $\mathrm{SL}_n(\mathbb{F}_q)$ is almost a simple finite group, *i.e.* for $q$ odd

$$1 \triangleleft \mathbb{Z}/2\mathbb{Z} \triangleleft \mathrm{SL}_n(\mathbb{F}_q)$$

with $\mathbb{Z}/2\mathbb{Z} = \{id, -id\} = \text{center}$. The quotient

$$\mathrm{SL}_n(\mathbb{F}_q)/\text{center}$$

is simple unless $q = 2, 3$

- Something similar works for all simple Lie groups, *e.g.* $\mathrm{SO}_n(\mathbb{F}_q)$ or $\mathrm{SP}_{2n}(\mathbb{F}_q)$, by work of Chevalley and Steinberg

## **Enter, the theorem!** (A short version of it.)

The discrete periodic table is complete:

(a) The cyclic groups $\mathbb{Z}/p\mathbb{Z}$ are simple for $p$ prime (infinitely many)

(b) The alternating groups $A_n$ are simple for $n \geq 5$ (infinitely many)

(c) There are 16 families of finite simple groups of Lie type (infinitely many)

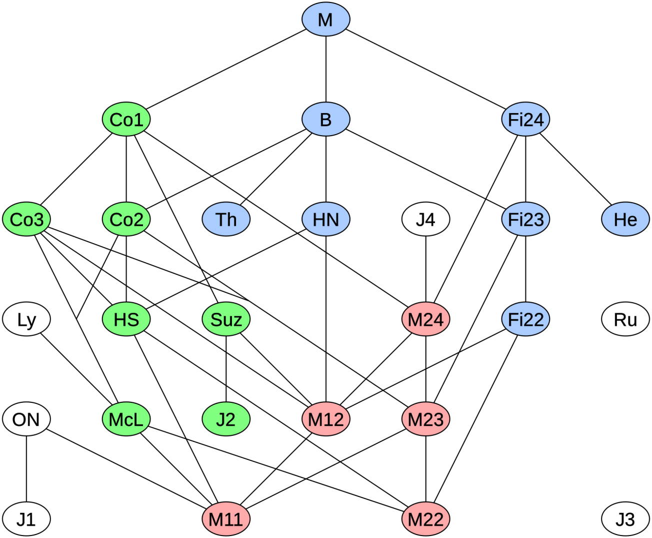
(d) There are 26 (or 27) exceptional finite simple groups (finitely many)

(e) There are no other simples This is the real meat

Note that almost all simple groups can be constructed using the smooth periodic table of simple Lie groups/Lie algebras

In some sense even the $A_n$ are of Lie type, using that "$S_n$ is $\mathrm{GL}_n(\mathbb{F}_1)$"

## Exotic discrete symmetries




The 26 sporadic simple groups correspond to “exotic symmetries” which have their vary unique place in nature

**Thank you for your attention!**

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I hope that was of some help.