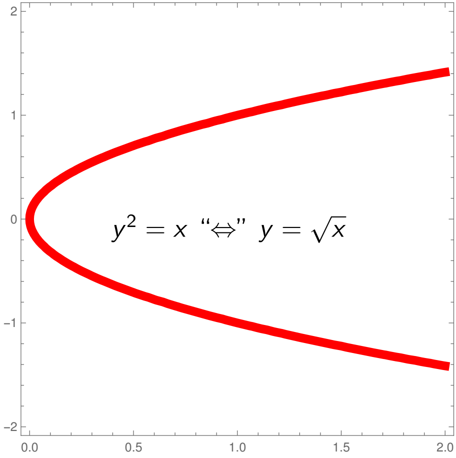
What are...Riemann surfaces?

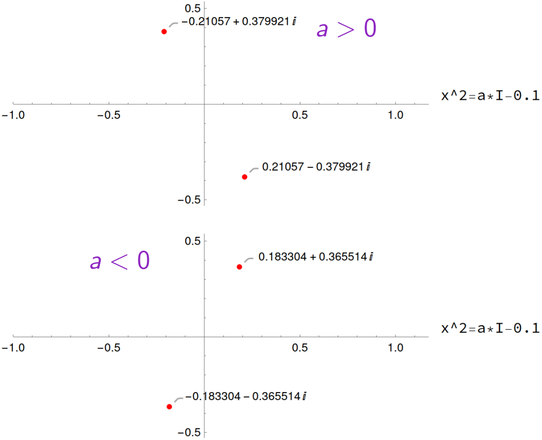
Or: Avoid choices!

Multi-valued functions



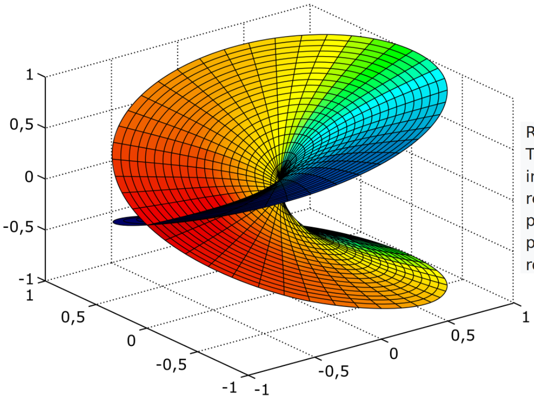
- The square root on $\mathbb{R}_{\geq 0}$ is a multi-valued function: there is some ambiguity defining it
- Most often, and a bit naively, the ambiguity is resolved by making choices
- Question How to avoid choices?

Complex square roots



- A choice for $\sqrt{\_}\colon \mathbb{C} \setminus \mathbb{R}_{<0} \to \mathbb{C}$ is to take the square root with positive real part
- Problem There is no continuous extension of $\sqrt{\_}$ over the missing half-line: when one approaches a point on the half-line from opposite sides, the limits of the chosen values differ by a sign

Avoid choices!



Riemann surface for the function $f(z) = \sqrt{z}$. The two horizontal axes represent the real and imaginary parts of z, while the vertical axis represents the real part of $\sqrt{z}$. The imaginary part of $\sqrt{z}$ is represented by the coloration of the points. For this function, it is also the height after rotating the plot 180° around the vertical axis.

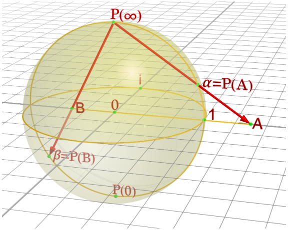
- $S = \{(z, w) \in \mathbb{C}^2 | w^2 = z\}$
- $\sqrt{z} = w$ is single-valued on S
- S is a first example of a Riemann surface (following history)

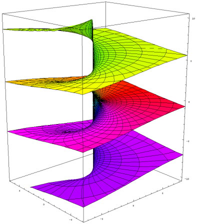
Enter, the theorem

Riemann surface = surface(=2d over $\mathbb{R}$) with a notion of complex-analytic functions

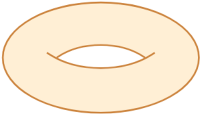
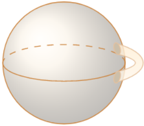
- **Theorem** The theory of compact Riemann surfaces is equivalent to that of projective algebraic curves They are manifolds and algebraic objects
- **Theorem** Every simply connected Riemann surface is equivalent to the Riemann sphere, $\mathbb{C}$ or upper half-plane (there are many more such results)

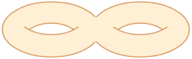
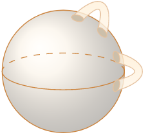
- Non-compact (right) Riemann surfaces are "designed" for multi-valued functions
- A compact (left) Riemann surface is "the opposite" of what I showed you so far

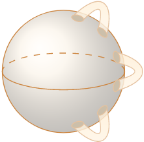
Sphere: 

, for log: 

Objects of complex analysis and topology

$S^2 \# \mathbb{T} \cong \mathbb{T} \cong$  $\cong$ 

$S^2 \# \#^2\mathbb{T} \cong \#^2\mathbb{T} \cong$  $\cong$ 

$S^2 \# \#^3\mathbb{T} \cong$  $\cong$ 

- Riemann surfaces can be divided into: hyperbolic, parabolic and elliptic
- Closed Riemann surfaces are classified by their genus = handles
- In contrast to plain surfaces, Riemann surfaces are always orientable

Thank you for your attention!

I hope that was of some help.