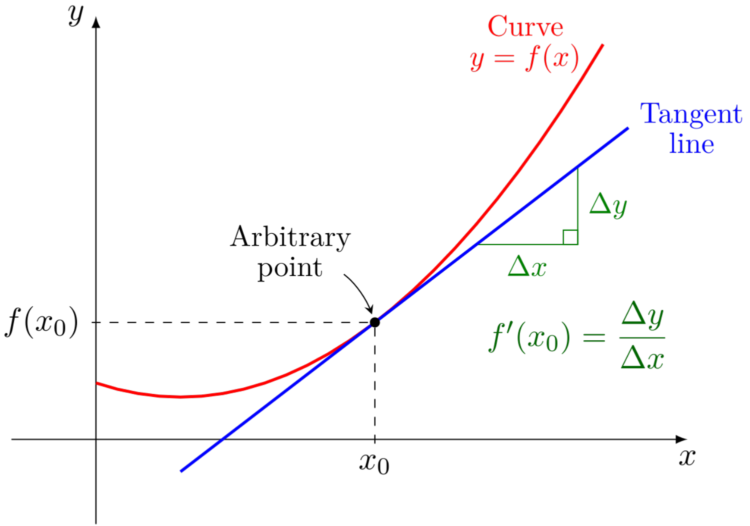
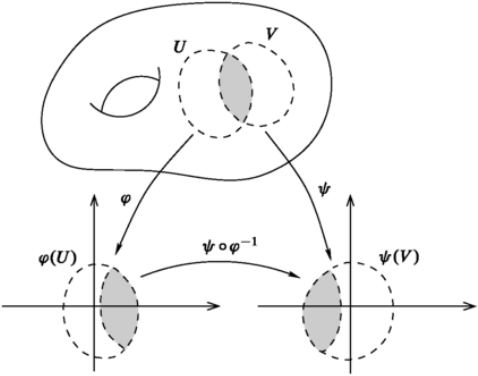
What is...exotic four space?

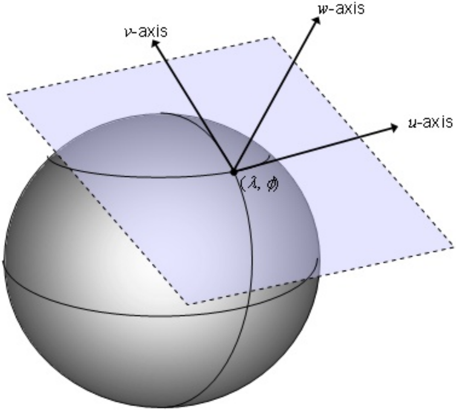
Or: Dimension four is weird!?



- **Calculus** is one of the main discovery of history
- **Question** How many generalizations of calculus exist?

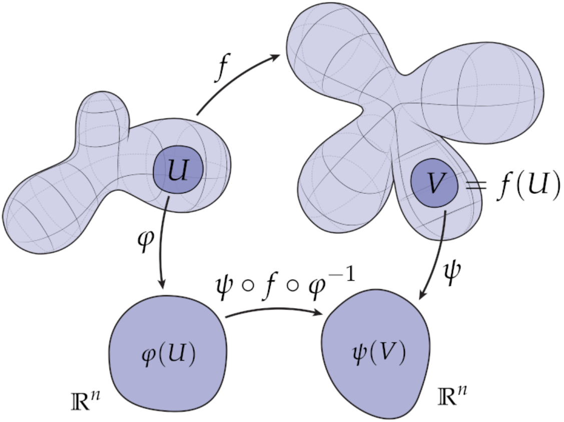
Calculus on manifolds





- Smooth n-manifold "$=$" something that locally looks like standard $\mathbb{R}^n$
- Being a smooth manifold involves a choice of a structure
- Examples $\mathbb{R}^n$, spheres S^n, more crazy stuff...
- Smooth manifolds allow calculus, *e.g.* Stokes theorem

Smooth structure



- $f: M \to N$ continuous is called smooth if $\phi \circ f \circ \psi^{-1}$ is smooth
- M and N have the same smooth structure if $\exists\, f: M \to N$ smooth + bijective + smooth inverse The correct notion of equivalence
- Question Are smooth structures unique, at least for "easy" manifolds?

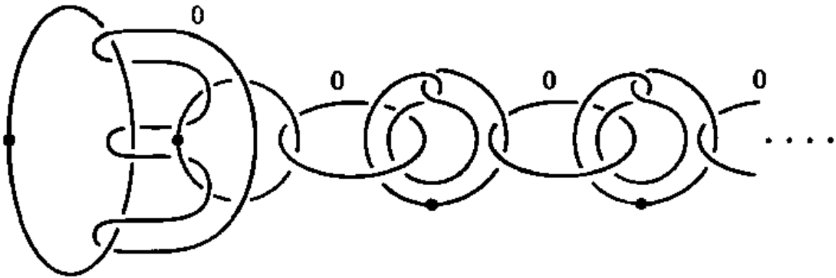
Enter, the theorems

For $\mathbb{R}^n$ we have:

(i) There exist a unique smooth structure on $\mathbb{R}^n$ for $n \neq 4$

(ii) There exist uncountably many smooth structures on $\mathbb{R}^4$ Exotic $\mathbb{R}^4$

- These can be constructed by surgery around knots



- $\exists$ exotic $\mathbb{R}^4$ that cannot be smoothly embedded into $\mathbb{R}^4$ "Very strange calculus"

And the sphere?

Dimension	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Smooth types	1	1	1	≥1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264	24

- Above: the number of different smooth structures on S^n; dim 4 $\leftrightsquigarrow$ 4d Poincaré conjecture
- Dim 4 is still weird We know "nothing" about smooth structures on S^4

Thank you for your attention!

I hope that was of some help.