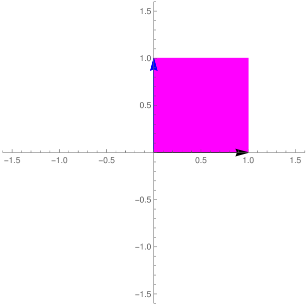
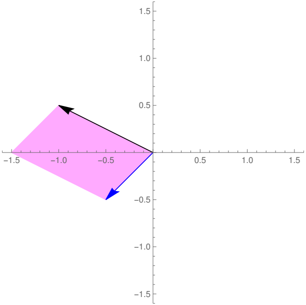
What is...the determinant?

Or: Signed permutations.

It is all about area



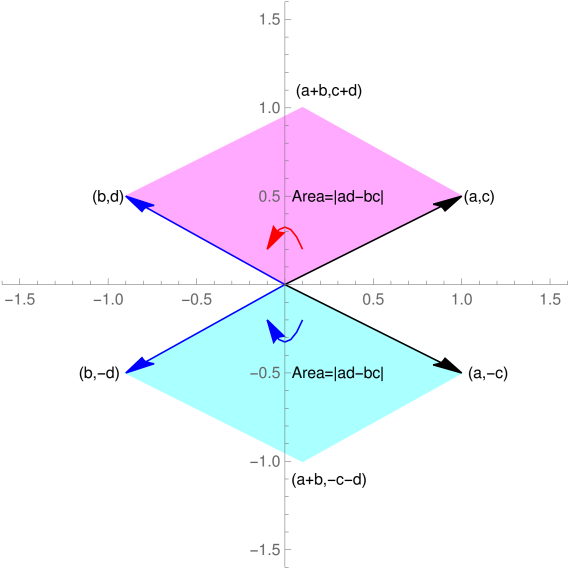
$$\xrightarrow{\begin{pmatrix} -1 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}}$$



Wish. The determinant should be the area of matrix times unit square.

It is all about area – well, almost

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

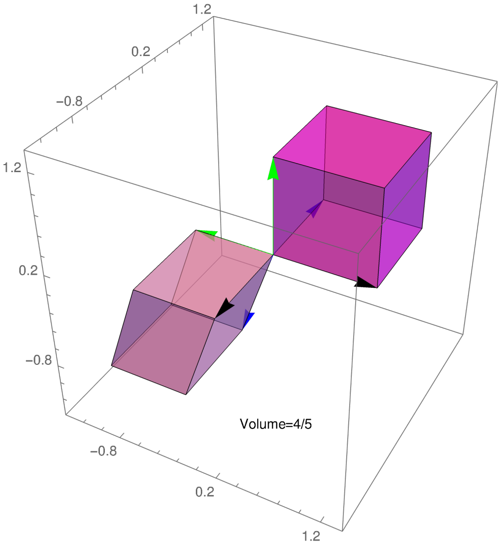


, $\begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$

So we let $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\det \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} = ad - bc$

What about higher dimensions?

$$\begin{pmatrix} 0 & -1/3 & -4/5 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$



Volume of matrix times unit cube is $\left|\det\begin{pmatrix} 0 & -1/3 & -4/5 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\right| = |-4/5|$

For completeness: A formal definition.

The determinant det is the unique function (non-trivial: it exists!) from *n*x*n* matrices to the ground field such that:

- $\det(id) = 1$
- det is multilinear on columns
- det is antisymmetric

The second bullet point is *e.g.*

$$\det\begin{pmatrix} a & b+e \\ c & d+f \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \det\begin{pmatrix} a & e \\ c & f \end{pmatrix}$$

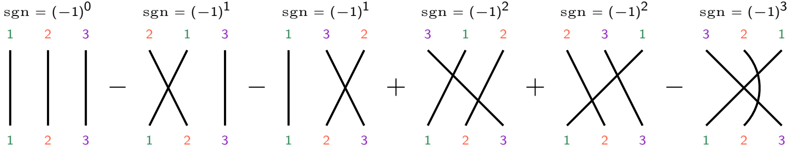
The third bullet point is *e.g.*

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\det\begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

Can you make this explicit?

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

Example. For $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$:

$\det(A) =$ 

$$= a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}$$

Thank you for your attention!

I hope that was of some help.