What are...direct sums \oplus ?

Or: How to add vector spaces and matrices.

My wish list for adding vector spaces.

▶ I want
$$V \oplus W \cong W \oplus V$$
.

- ▶ I want $(V \oplus W) \oplus X \cong V \oplus (W \oplus X)$.
- ▶ I want dim $(V \oplus W) = \dim(V) + \dim(W)$.

Does this remind you of numbers?

How can we add vectors externally?

$$\mathbb{R}^{3} \oplus \mathbb{R}^{2} \ni \begin{pmatrix} 1\\2\\3 \end{pmatrix} \oplus \begin{pmatrix} 4\\5 \end{pmatrix} = \begin{pmatrix} 1\\2\\3\\4\\5 \end{pmatrix} \quad \iff \quad \begin{pmatrix} 1\\2\\3\\4\\5 \end{pmatrix} \in \mathbb{R}^{5}$$

Vector spaces V and W have bases $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_m\}$. The space $V \oplus W$ has bases $\{(v_1, 0), ..., (v_n, 0), (0, w_1), ..., (0, w_m)\}$.

Two bases give add to a new one:

$$\left\{ \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\\0\\0\\1 \end{pmatrix} \right\} \quad \iff \quad \left\{ \begin{pmatrix} 1\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0\\0\\1 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 1\\0\\0\\0\\1 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 1\\0\\0\\0\\1 \end{pmatrix} \right\}$$

$$V \oplus W \cong W \oplus V? \text{ Yep:}$$

$$\begin{pmatrix} 1\\2\\3 \end{pmatrix} \oplus \begin{pmatrix} 4\\5 \end{pmatrix} = \begin{pmatrix} 1\\2\\3\\4\\5 \end{pmatrix} = " \begin{pmatrix} 4\\5\\1\\2\\3 \end{pmatrix} = \begin{pmatrix} 4\\5 \end{pmatrix} \oplus \begin{pmatrix} 1\\2\\3 \end{pmatrix}$$

$$(V \oplus W) \oplus X \cong V \oplus (W \oplus X)?$$
 Yep:
$$\left(\begin{pmatrix} 1\\2\\3 \end{pmatrix} \oplus \begin{pmatrix} 4\\5 \end{pmatrix} \right) \oplus (6) = \underbrace{\begin{pmatrix} 1\\2\\3\\4\\5 \end{pmatrix}}_{6} = \begin{bmatrix} 1\\2\\3\\4\\5\\6 \end{pmatrix} = \begin{bmatrix} 1\\2\\3\\4\\5\\6 \end{pmatrix} = \begin{pmatrix} 1\\2\\3\\4 \end{bmatrix} \oplus \begin{pmatrix} 4\\5 \end{pmatrix} \oplus (6) \right)$$

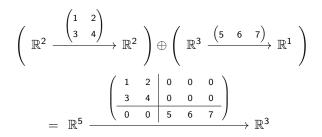
 $\dim(V \oplus W) = \dim(V) + \dim(W)$? Yep:

$$\#\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}\right\} = 5 = 3 + 2 = \#\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\} + \#\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

For completeness: A formal definition.

If V and W are vector spaces, then $V \oplus W$ is the vector space whose: elements are pairs (v, w) with $v \in V$ and $w \in W$; addition is componentwise, *i.e.* (v, w) + (v', w') = (v + v', w + w'); scalar multiplication is componentwise, *i.e.* $\lambda(v, w) = (\lambda v, \lambda w)$.

And what about matrices?



They really do not know each other ;-)

Thank you for your attention!

I hope that was of some help.