What are...tensor products $\otimes$?

Or: How to multiply vector spaces and matrices.

My wish list for multiplying vector spaces.

- I want $V \otimes W \cong W \otimes V$.
- I want $(V \otimes W) \otimes X \cong V \otimes (W \otimes X)$.
- I want $\dim(V \otimes W) = \dim(V) \dim(W)$.

Does this remind you of numbers?

How can we multiply vectors externally?

$$\mathbb{R}^3 \otimes \mathbb{R}^2 \ni \begin{pmatrix}1\\2\\3\end{pmatrix} \otimes \begin{pmatrix}4\\5\end{pmatrix} = \begin{array}{c|cc} \otimes & 4 & 5 \\ \hline 1 & 1\cdot4 & 1\cdot5 \\ 2 & 2\cdot4 & 2\cdot5 \\ 3 & 3\cdot4 & 3\cdot5 \end{array} = \left(\begin{array}{c} 1\cdot4 \\ 1\cdot5 \\ \hline 2\cdot4 \\ 2\cdot5 \\ \hline 3\cdot4 \\ 3\cdot5 \end{array}\right) \quad \leftrightsquigarrow \quad \begin{pmatrix}4\\5\\8\\10\\12\\15\end{pmatrix} \in \mathbb{R}^6$$

Vector spaces V and W have bases $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_m\}$.
The space $V \otimes W$ has bases $\{v_1 w_1, ..., v_1 w_m, ..., v_n w_m\}$.

Two bases multiply to a new one:

$$\left\{ \begin{array}{cc} \begin{array}{c|cc} \otimes & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}, & \begin{array}{c|cc} \otimes & 0 & 1 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \\ \begin{array}{c|cc} \otimes & 1 & 0 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}, & \begin{array}{c|cc} \otimes & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{array} \\ \begin{array}{c|cc} \otimes & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}, & \begin{array}{c|cc} \otimes & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \end{array} \right\} \quad \leftrightsquigarrow \quad \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \otimes \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

Where my wishes granted?

$V \otimes W \cong W \otimes V$? Yep:

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \otimes \begin{pmatrix}4\\5\end{pmatrix} = \begin{array}{c|cc} \otimes & 4 & 5 \\ \hline 1 & 4 & 5 \\ 2 & 8 & 10 \\ 3 & 12 & 15 \end{array} \text{“=”} \begin{array}{c|ccc} \otimes & 1 & 2 & 3 \\ \hline 4 & 4 & 8 & 12 \\ 5 & 5 & 10 & 15 \end{array} = \begin{pmatrix}4\\5\end{pmatrix} \otimes \begin{pmatrix}1\\2\\3\end{pmatrix}$$

$(V \otimes W) \otimes X \cong V \otimes (W \otimes X)$? Yep:

$$\left(\begin{pmatrix}1\\2\\3\end{pmatrix} \otimes \begin{pmatrix}4\\5\end{pmatrix}\right) \otimes (6) = \begin{pmatrix}(1\cdot4)\cdot6\\(1\cdot5)\cdot6\\ \hline (2\cdot4)\cdot6\\(2\cdot5)\cdot6\\ \hline (3\cdot4)\cdot6\\(3\cdot5)\cdot6\end{pmatrix} \text{“=”} \begin{pmatrix}1\cdot(4\cdot6)\\1\cdot(5\cdot6)\\ \hline 2\cdot(4\cdot6)\\2\cdot(5\cdot6)\\ \hline 3\cdot(4\cdot6)\\3\cdot(5\cdot6)\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \otimes \left(\begin{pmatrix}4\\5\end{pmatrix} \otimes (6)\right)$$

$\dim(V \otimes W) = \dim(V)\dim(W)$? Yep:

$$\#\left\{\begin{array}{c} \begin{array}{c|cc} \otimes & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}, \begin{array}{c|cc} \otimes & 0 & 1 \\ \hline 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \\ \begin{array}{c|cc} \otimes & 1 & 0 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}, \begin{array}{c|cc} \otimes & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{array} \\ \begin{array}{c|cc} \otimes & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}, \begin{array}{c|cc} \otimes & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \end{array}\right\} = 6 = 3 \cdot 2 = \#\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} + \#\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

For completeness: A formal definition.

If V and W are vector spaces with fixed standard bases $\{e_i\}$, $\{e_j\}$, then $V \otimes W$ is the vector space defined by demanding that

$e_1 \otimes e_1 = e_{11} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$, ..., $e_n \otimes e_m = e_{nm} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 1 \end{pmatrix}$ is the new basis of $V \otimes W$.

And what about matrices?

$$\left(\mathbb{R}^2 \xrightarrow{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \mathbb{R}^2 \right) \otimes \left(\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 5 & 6 & 7 \end{pmatrix}} \mathbb{R}^1 \right)$$

$$= \mathbb{R}^6 \xrightarrow{\left(\begin{array}{cc|cc|cc} 5 & 10 & 6 & 12 & 7 & 14 \\ 15 & 29 & 18 & 24 & 21 & 28 \end{array}\right)} \mathbb{R}^2$$

They are intertwined in a multiplicative way.

Thank you for your attention!

I hope that was of some help.