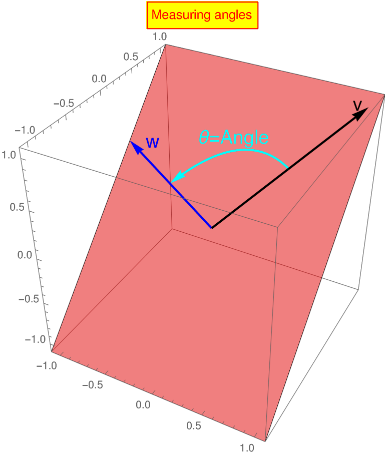
What is...an inner product?

Or: Generalizing angles and lengths.

The dot product

$$\langle v, w\rangle = \Big\langle (1, 1, 1), (-1, 1/4, 1/2) \Big\rangle = 1 \cdot -1 + 1 \cdot 1/4 + 1 \cdot 1/2 = -1/4$$

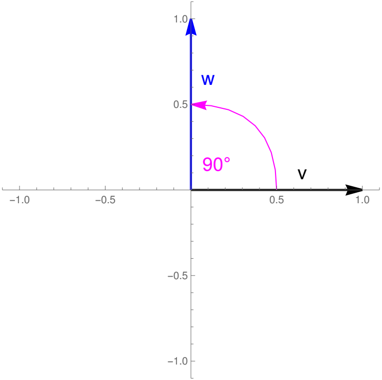
$$\mathrm{Length}(v) = \sqrt{\langle v, v\rangle} = \sqrt{3}, \quad \mathrm{Length}(w) = \sqrt{\langle w, w\rangle} = \sqrt{21}/4$$



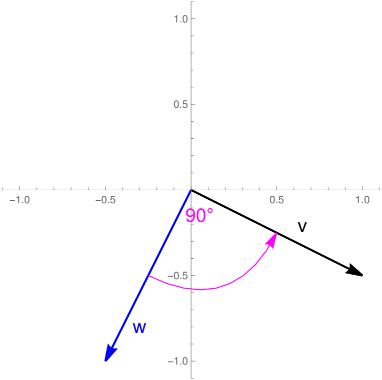
$$\mathrm{Length}(v) \cdot \mathrm{Length}(w) \cdot \cos(\Theta)$$
$$= -1/4$$
$$\rightsquigarrow$$
$$\Theta = \arccos\big(-1/(3\sqrt{7})\big)$$

Orthogonality

$v = (1, 0)$, $w = (0, 1)$



$v = (1, -1/2)$, $w = (-1/2, -1)$



v, w are orthogonal if and only if $\langle v, w \rangle = 0$

Why not take this as a definition ?

The dot product is linear, symmetric and positive definite

$$\langle v, w\rangle = v_1 w_1 + v_2 w_2 + v_3 w_3$$

(a) $\langle \lambda v, w\rangle = \lambda v_1 w_1 + \lambda v_2 w_2 + \lambda v_3 w_3 = \lambda\langle v, w\rangle$

(b) $\langle v + v', w\rangle = (v_1 + v_1')w_1 + (v_2 + v_2')w_2 + (v_3 + v_3')w_3 = \langle v, w\rangle + \langle v', w\rangle$

(c) $\langle v, w\rangle = v_1 w_1 + v_2 w_2 + v_3 w_3 = \langle w, v\rangle$

(d) $\langle v, v\rangle = \text{Length}(v) > 0$ for $v \neq 0$

(a) and (b) are clear by

$$\langle v, w\rangle = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

For completeness: A formal definition.

An inner product is a map

$$\langle \_, \_ \rangle \colon V \times V \to \mathbb{R}$$

satisfying:

(a) $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$ (Linearity 1)

(b) $\langle v + v', w \rangle = \langle v, w \rangle + \langle v', w \rangle$ (Linearity 2)

(c) $\langle v, w \rangle = \langle w, v \rangle$ (Symmetry)

(d) $\langle v, v \rangle > 0$ for $v \neq 0$ (Positive definite)

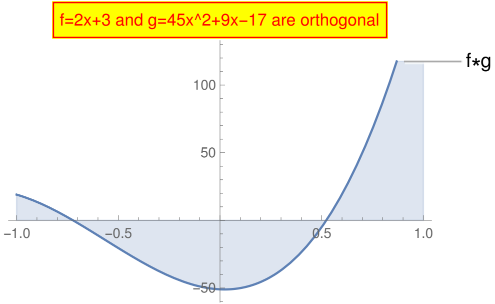
A word of warning: Over $\mathbb{C}$ the condition $\langle v, w \rangle = \langle w, v \rangle$ is replaced by

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

We can now talk about angles, length, orthogonality for...

- ...vectors in $\mathbb{R}^n$ via $\langle v, w \rangle = v_1 w_1 + ... + v_n w_n$
- ...nxn matrices via $\langle M, N \rangle = \mathrm{tr}(MN^T)$
- ...(reasonable) functions $f\colon [0,1] \to \mathbb{R}$ via $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$
- ...more

Orthogonality could mean that areas cancel, *e.g.*:



Thank you for your attention!

I hope that was of some help.