What is...the exponential of a matrix?

Or: Nilpotent vanishing.

The exponential function

The exponential function

$$e^m = 1 + \frac{1}{1!}m^1 + \frac{1}{2!}m^2 + \frac{1}{3!}m^3 + \ldots$$

is ubiquitous in mathematics. Can we generalize it?

It formally makes sense for any m in some real vector space as long as you can multiply m. In this setting we have the classical properties, *e.g.*:

- We have
$$e^0 = 1$$
- We have
$$e^{p^{-1}mp} = p^{-1}e^m p$$
- We have
$$e^{d+n} = e^d e^n$$

  This proof uses $dn = nd$.

Let us look at Jordan blocks

$$M = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

- M is a diagonal D plus a nilpotent matrix N:

$$M = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = D + N = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- We have

$$e^D = id + \frac{1}{1!}\begin{pmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^1 & 0 \\ 0 & 0 & \lambda^1 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} + \ldots = \begin{pmatrix} e^\lambda & 0 & 0 \\ 0 & e^\lambda & 0 \\ 0 & 0 & e^\lambda \end{pmatrix}$$

- We have

$$e^N = id + \frac{1}{1!}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \ldots = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- Since $DN = ND$, we have

$$e^M = e^{D+N} = e^D e^N = \begin{pmatrix} e^\lambda & e^\lambda & e^\lambda/2 \\ 0 & e^\lambda & e^\lambda \\ 0 & 0 & e^\lambda \end{pmatrix} = e^\lambda e^N$$

Wait: This always works?

$$M = P^{-1}(D+N)P$$

$$e^M = e^{P^{-1}(D+N)P} = P^{-1}e^{(D+N)}P = P^{-1}e^De^NP$$

since $DN = ND$ always holds.

Example.

$$M = \begin{pmatrix} 2 & -1/2 & -1/2 \\ 0 & 3/2 & -1/2 \\ -1 & 1/2 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

Thus:

$$e^M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} e^1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} e^1 & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

For completeness: A formal definition.

Let M be an nxn real or complex matrix. The exponential e^M of M is the nxn matrix given by the power series

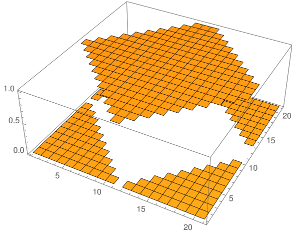
$$e^M = M^0 + \frac{1}{1!}M^1 + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + ...$$

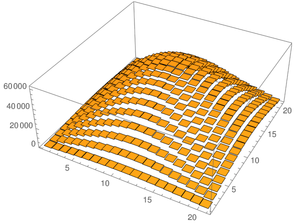
where M^0 is the nxn identity matrix.

The series always converges and for $n = 1$ one recovers the classical exponential function.

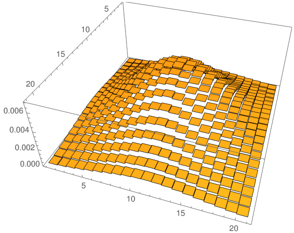
Here come some funny examples.

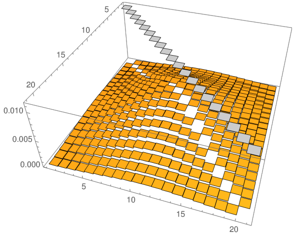
Diamond and Exp(Diamond)=Gaussian:





Gaussian and Exp(Gaussian)=Crossed Gaussian:





Thank you for your attention!

I hope that was of some help.