What is...the Jordan normal form again?

Or: How to find and compute a gold standard.

An almost Jordan block

$$M = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

The only eigenvector is $(1, 0, 0)$.

$$N = M - \lambda id = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Still the same eigenvectors.

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

New "eigenvector" $(0, 1, 0)$.

$$N^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

New "eigenvector" $(0, 0, 1)$.

This calculation does not depend on λ or on 1.

Why not take 0? We get Jordan blocks.

An example

$$M = \begin{pmatrix} -5/2 & 1 & 1 & 3/2 \\ -9/2 & 3 & 1 & 3/2 \\ -13/2 & 1 & 4 & 5/2 \\ -11/2 & 1 & 1 & 9/2 \end{pmatrix}$$

- First, we calculate the characteristic polynomial. We get $p(X) = (X-1)(X-2)(X-3)^2$.
- An eigenvector for $\lambda = 1$ is $(1,1,1,1)$. An eigenvector for $\lambda = 2$ is $(1,2,1,1)$. An eigenvector for $\lambda = 3$ is $(1,1,3,1)$.
- Form a base-change matrix

$$P = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

- We get a simpler matrix, which is almost there:

$$P^{-1}MP = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1/2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

This is where the kernels come in!

$$M - 3id = \begin{pmatrix} -11/2 & 1 & 1 & 3/2 \\ -9/2 & 0 & 1 & 3/2 \\ -13/2 & 1 & 1 & 5/2 \\ -11/2 & 1 & 1 & 3/2 \end{pmatrix}$$

$$(M - 3id)^2 = \begin{pmatrix} 11 & -3 & -2 & -2 \\ 10 & -2 & -2 & -2 \\ 11 & -3 & -2 & -2 \\ 11 & -3 & -2 & -2 \end{pmatrix}$$

has the new, interesting eigenvector $(1, 1, 1, 3)$ in its kernel.

Indeed:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} -5/2 & 1 & 1 & 3/2 \\ -9/2 & 3 & 1 & 3/2 \\ -13/2 & 1 & 4 & 5/2 \\ -11/2 & 1 & 1 & 9/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

For completeness: A formal definition.

A generalized eigenvector v_m of M is a solution to

$$(M - \lambda id)^m v_m = 0$$

for λ an eigenvalue. It is of rank m if

$$(M - \lambda id)^{m-1} v_m \neq 0$$

A chain $\{v_m, ..., v_1\}$ of generalized eigenvectors is a linear independent set satisfying

$$v_j = (M - \lambda id)^{m-j} v_m$$

Such a chain is called a Jordan chain if m is maximal with respect to those properties.

For every complex matrix M there is a basis made of Jordan chains.

Well, almost. Warning!

$$\begin{pmatrix} 1/2 & 0 & 1/2 \\ -1 & 1 & 1 \\ -1/2 & 0 & 3/2 \end{pmatrix}$$

Eigenvectors $(1,0,1)$ and $(0,1,0)$.

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 1 & 0 & c \end{pmatrix}^{-1} \begin{pmatrix} 1/2 & 0 & 1/2 \\ -1 & 1 & 1 \\ -1/2 & 0 & 3/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 1 & 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1/2(-a+c) \\ 0 & 1 & -a+c \\ 0 & 0 & 1 \end{pmatrix}$$

We can not go on from here since $\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 1 & 0 & a \end{pmatrix}$ is not invertible.

We need to find a Jordan chain. In this case we can use:

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_1' = \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix}, \quad v_2' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 & 0 & 1/2 \\ -1 & 0 & 1 \\ -1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix}$$

Thank you for your attention!

I hope that was of some help.