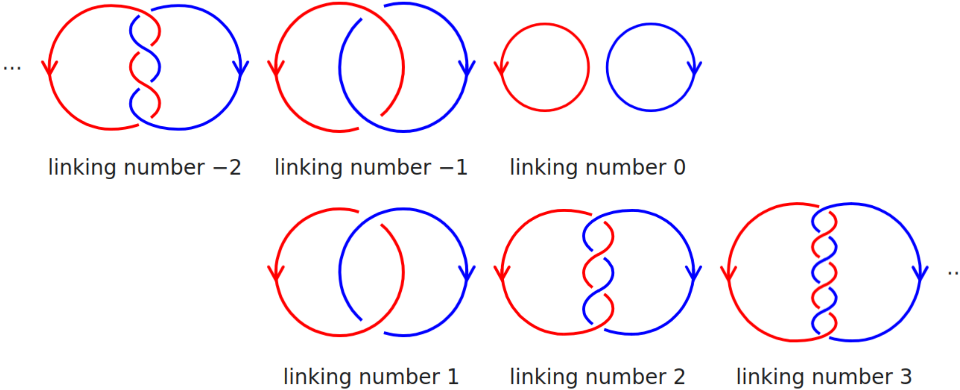
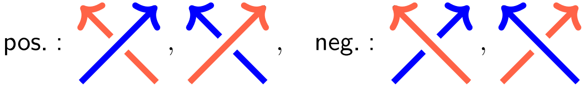
What is...the linking matrix?

Or: Kirby calculus on matrices

Gauss' linking number

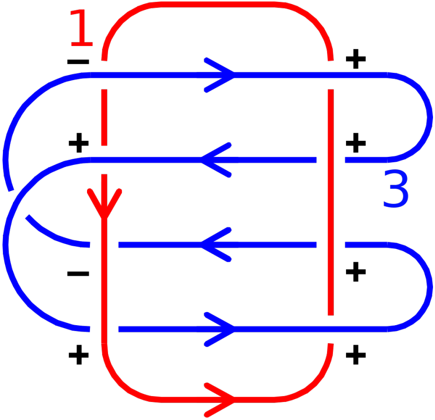


- Positive and negative crossings



- Linking number $lk(K_1, K_2) = \frac{1}{2}$(# of positive − # of negative crossings)

Linking matrix

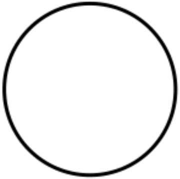


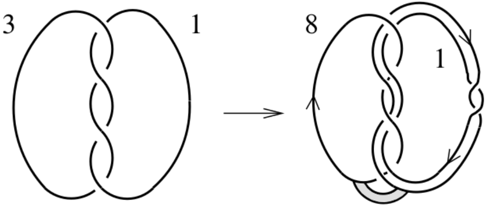
$$\leftrightarrow \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

- Fix K_1 to K_n, consider them as framed by $f_i \in \mathbb{Z}$
- **Linking matrix $A = (a_{ij})_{i,j}$** is the nxn matrix with

$$a_{ij} = \begin{cases} f_i & \text{if } i = j \\ lk(K_i, K_j) & \text{if } i \neq j \end{cases}$$

Kirby and linking

$\mathscr{L} \mapsto \mathscr{L} \cup$  $^{\pm 1}$ $\rightsquigarrow$ $\begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \pm 1 \end{pmatrix}$



$\rightsquigarrow \begin{pmatrix} 3 & \pm 2 \\ \pm 2 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 8 & 3 \\ 3 & 1 \end{pmatrix}$

- The Kirby move I changes A by adding an ± 1 entry
- The Kirby move II changes A by adding (or subtracting) the jth row to (from) the ith row and the jth column to (from) the ith column

For completeness: A formal statement

The linking matrix A is...

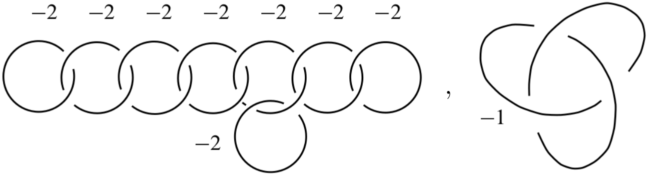
(i) ...an invariant of the K_1 to K_n seen as framed

(ii) ...an invariant of the 3mfd up to the matrix Kirby moves:

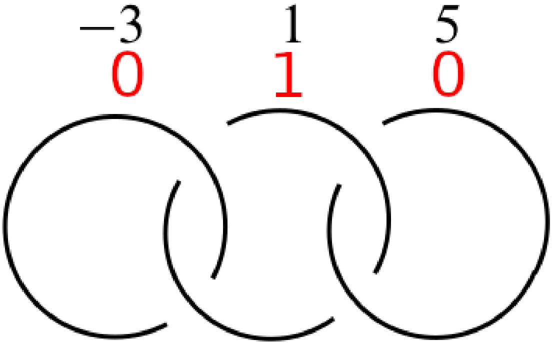
- The Kirby move I changes A by adding an ± 1 entry
- The Kirby move II changes A by adding (or subtracting) the jth row to (from) the ith row and the jth column to (from) the ith column

A can be used to prove that every closed, orientable, connected 3mfd can be obtained by Dehn surgery with only even framing

same 3mfd:



Getting rid of odd bits



$$\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

- Take $B = A \bmod 2$ and solve $B.(x_1, ..., x_n) = (b_{11}, ..., b_{nn})$ Linear algebra
- Solutions are called characteristic subknots
- Getting rid of these via Kirby moves shows the "even theorem"

Thank you for your attention!

I hope that was of some help.