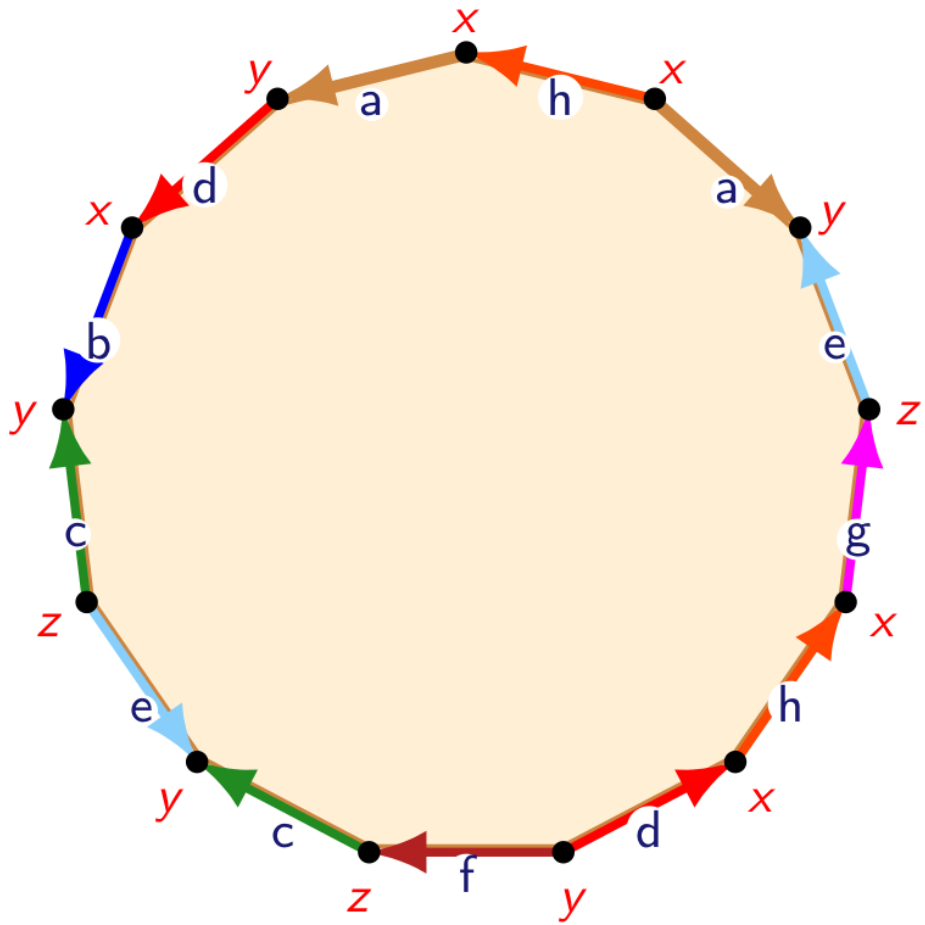
What are...words for surfaces?

Or: How to “read” surfaces

From a polygon to a word



$adb\bar{c}e\bar{c}\bar{f}\,dhge\bar{a}h$

- A polygon as above $\leftrightsquigarrow$ a surface
- Reading counterclockwise gives the surface word

Words on a necklace

abc

$=$

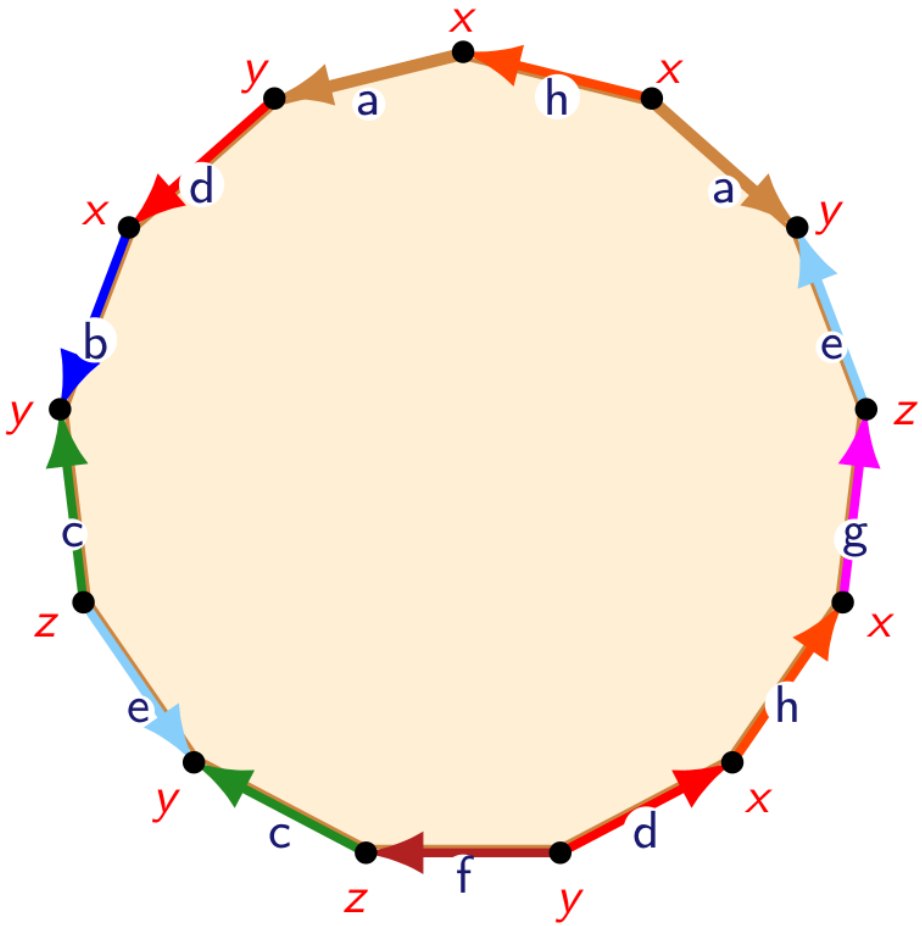
cab

$=$

bca

- The starting point for the reading should not give different surfaces
- Words give the same surface if they are related by a cyclic permutation
- Said differently, words live on a necklace

A few more relations



Free edges
$x \quad b \quad y$
$x \quad g \quad z$
$y \quad f \quad z$

$\leftrightsquigarrow$ bgf = one edge

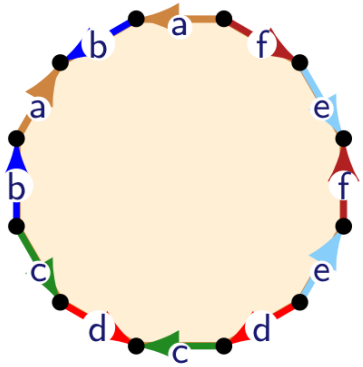
- Non paired edges in a row can be contracted, *e.g.* $abc = a$ for non paired a, b, c
- "In a row" is meant for the same vertices
- The vertices can be spread over the word

For completeness: A formal statement

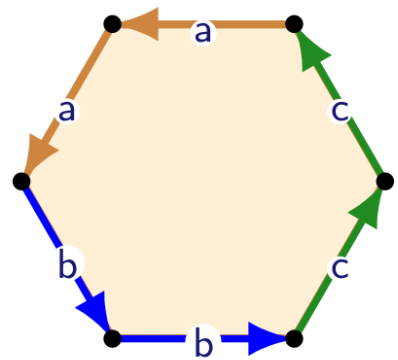
Every closed surface $S \neq S^2$ is of the form

$$S \cong (\#^h T)\#(\#^p \mathbb{R}P^2)$$

The standard words for closed surfaces are

$\#^3\mathbb{T} =$ 

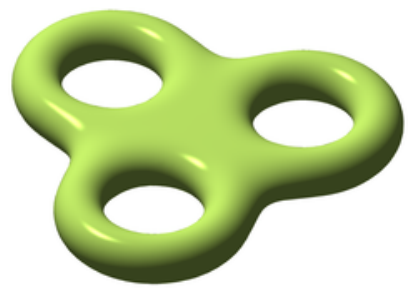
$= a\ b\ \overline{a}\ \overline{b}\ c\ d\ \overline{c}\ \overline{d}\ e\ f\ \overline{e}\ \overline{f}$

$\#^3\mathbb{P}^2 =$ 

$= a\ a\ b\ b\ c\ c$

$\ldots \quad \#^t\mathbb{T} = a_1\ b_1\ \overline{a_1}\ \overline{b_1}\ a_2\ b_2\ \overline{a_2}\ \overline{b_2}\ \ldots\ a_t\ b_t\overline{a_t}\ \overline{b_t}$

$\ldots \quad \#^p\mathbb{P}^2 = a_1\ a_1\ a_2\ a_2\ \ldots\ a_p\ a_p$

$\#^3\mathbb{T} =$ 

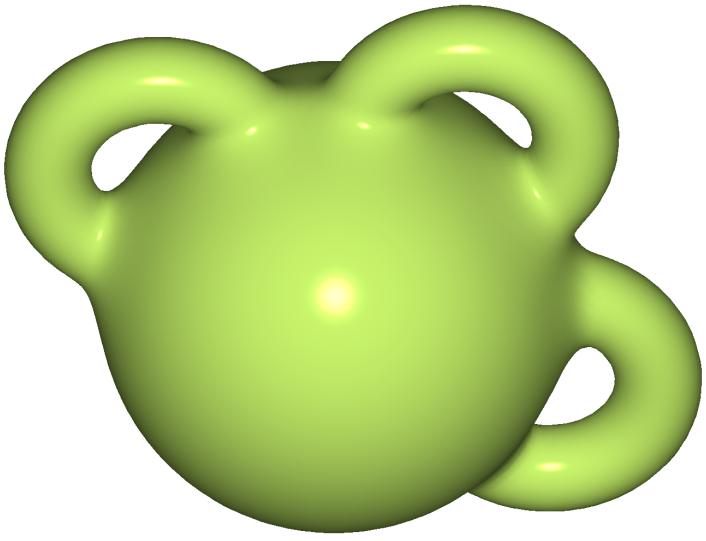
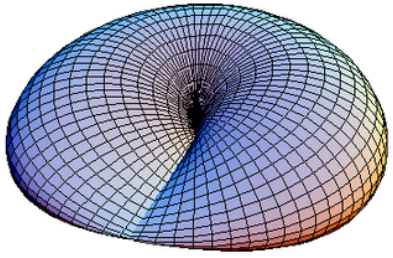
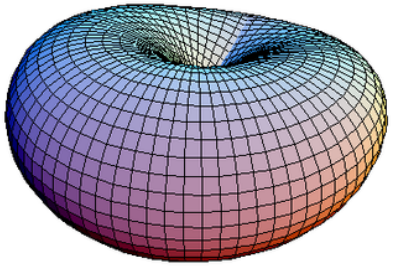
For non-closed surfaces use the same and the previous relations

The general classification

Every surface S is of the form

$$S \cong S^2 \# (\#^d D) \# (\#^h T) \# (\#^p \mathbb{R}P^2)$$

d punctures, h handles, p projective planes

From left to right:

- A sphere with a puncture
- A sphere with three handles
- A sphere with a projective plane

Thank you for your attention!

I hope that was of some help.