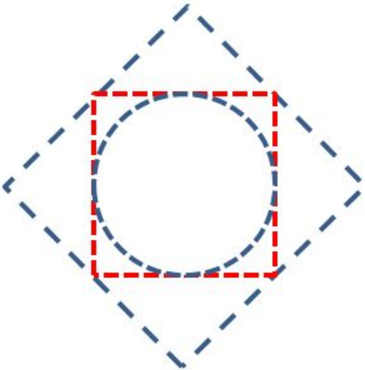
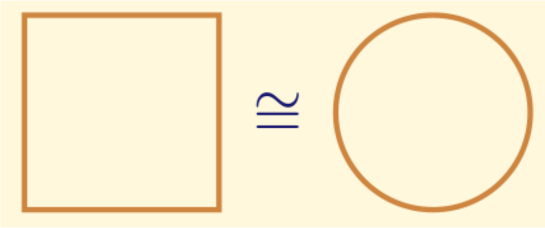
What are...homeomorphisms?

Or: Rubber geometry

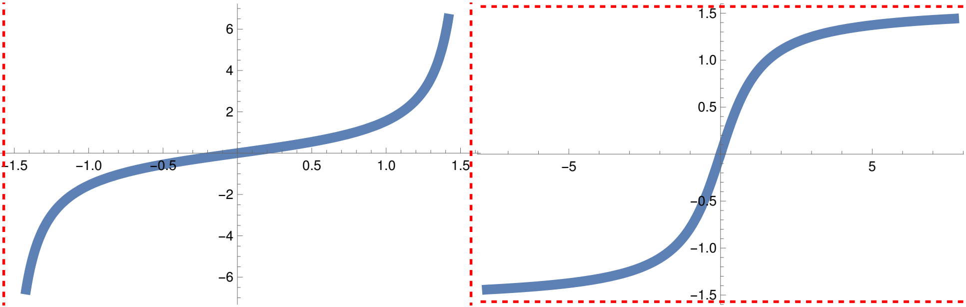
The square is the circle





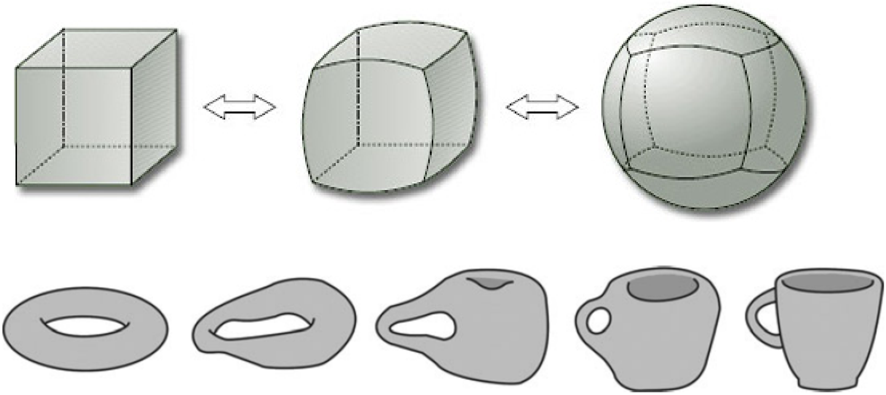
- A continuous map from the square to the circle: $(x, y) \mapsto (x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2})$
- A continuous map from the circle to the square: $(x, y) \mapsto (x/|x^2 + y^2|, y/|x^2 + y^2|)$
- These maps are inverses
- Hence, from the viewpoint of topology square $=$ circle

An interval is a line



- A continuous map from $(-\pi/2, \pi/2)$ to $\mathbb{R}$: $x \mapsto \tan(x)$
- A continuous map from $\mathbb{R}$ to $(-\pi/2, \pi/2)$: $x \mapsto \arctan(x)$
- These maps are inverses
- Hence, from the viewpoint of topology $(-\pi/2, \pi/2) = \mathbb{R}$

Like rubber



- Idea In topology one identifies spaces up to continuous-rubber deformations
- What is not allowed is tearing
- Writing down explicit maps is unpractical and one usually does not do it

For completeness: A formal definition

Two topological spaces X and Y are homeomorphic if:

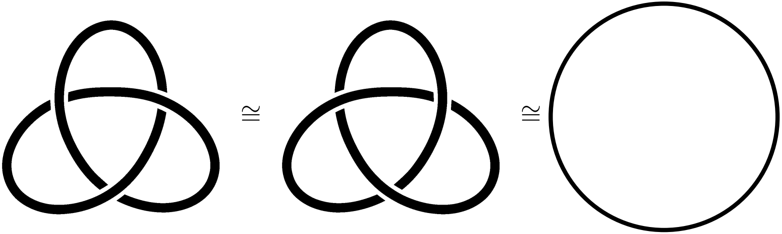
(i) There exists a continuous map $f\colon X \to Y$

(ii) There exists a continuous map $g\colon Y \to X$

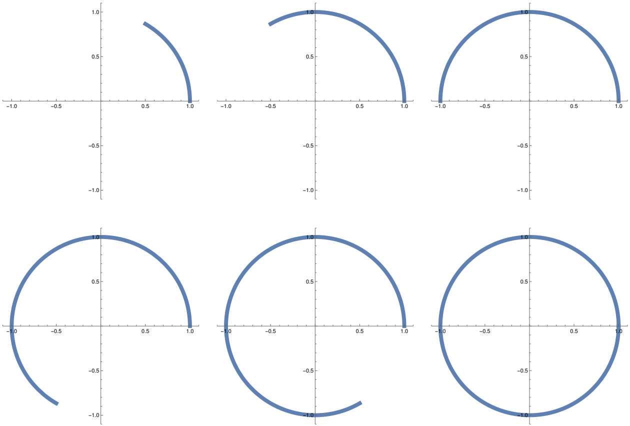
(iii) These maps are inverses

We write $X \cong Y$ for homeomorphic spaces

- Homeomorphism is the most used equivalence relation in topology
- But sometimes one needs different notions, as *e.g.* homeomorphisms do not detect embeddings:



The circle is not an interval



- $x \mapsto (\cos(x), \sin(x))$ is a continuous bijection from $[0, 2\pi)$ to the circle
- The inverse is not continuous
- The circle is not homeomorphic to an interval (strictly speaking we need some invariant to check that)

Thank you for your attention!

I hope that was of some help.