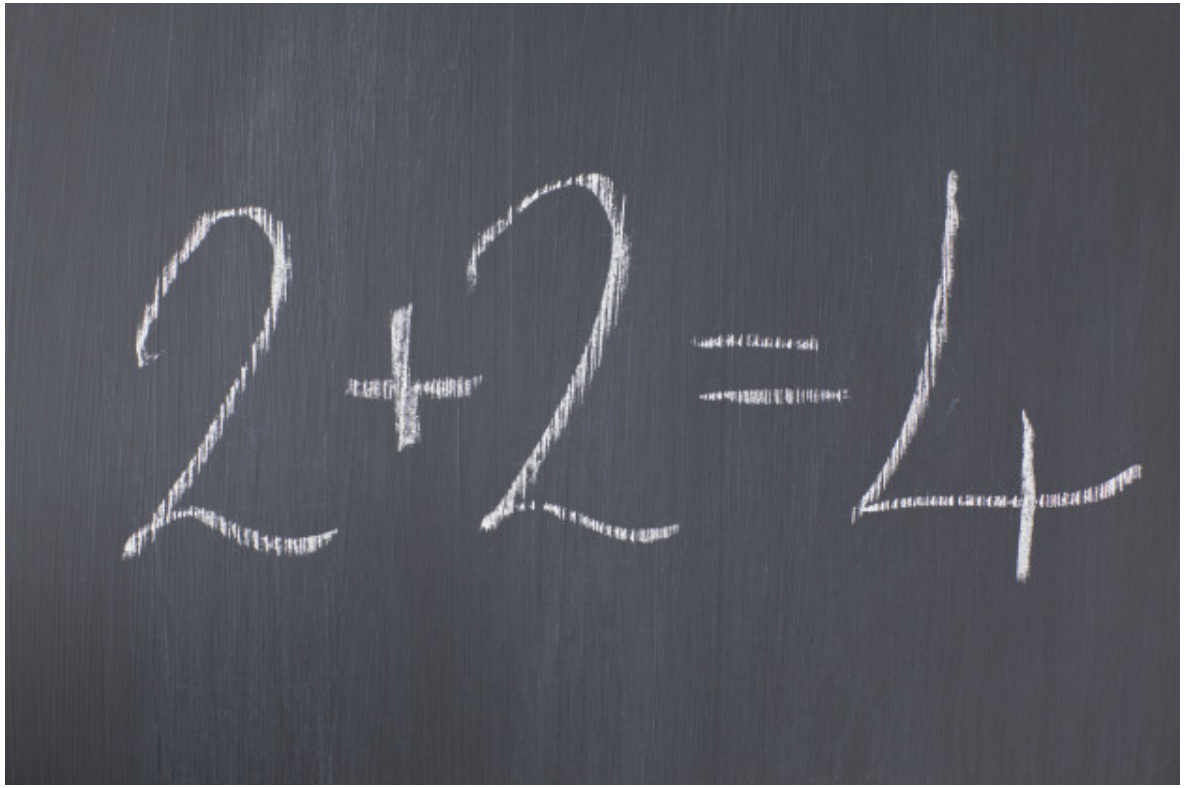
What is...equivalence?

Or: Equal means...?

The science of equality!?



- In some sense mathematics is the science of equality

$$\text{real world: } \pi = 3, \quad \text{maths: } \pi = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$$

- In some other sense mathematics is the science of redefining equality

$$\pi \approx 3, \quad \pi = 3.14..., \quad \text{many more}$$

The science of equivalence!

$\mathbb{Z}/2\mathbb{Z}$:

$+$	0	1
0	0	1
1	1	0

,

FG:

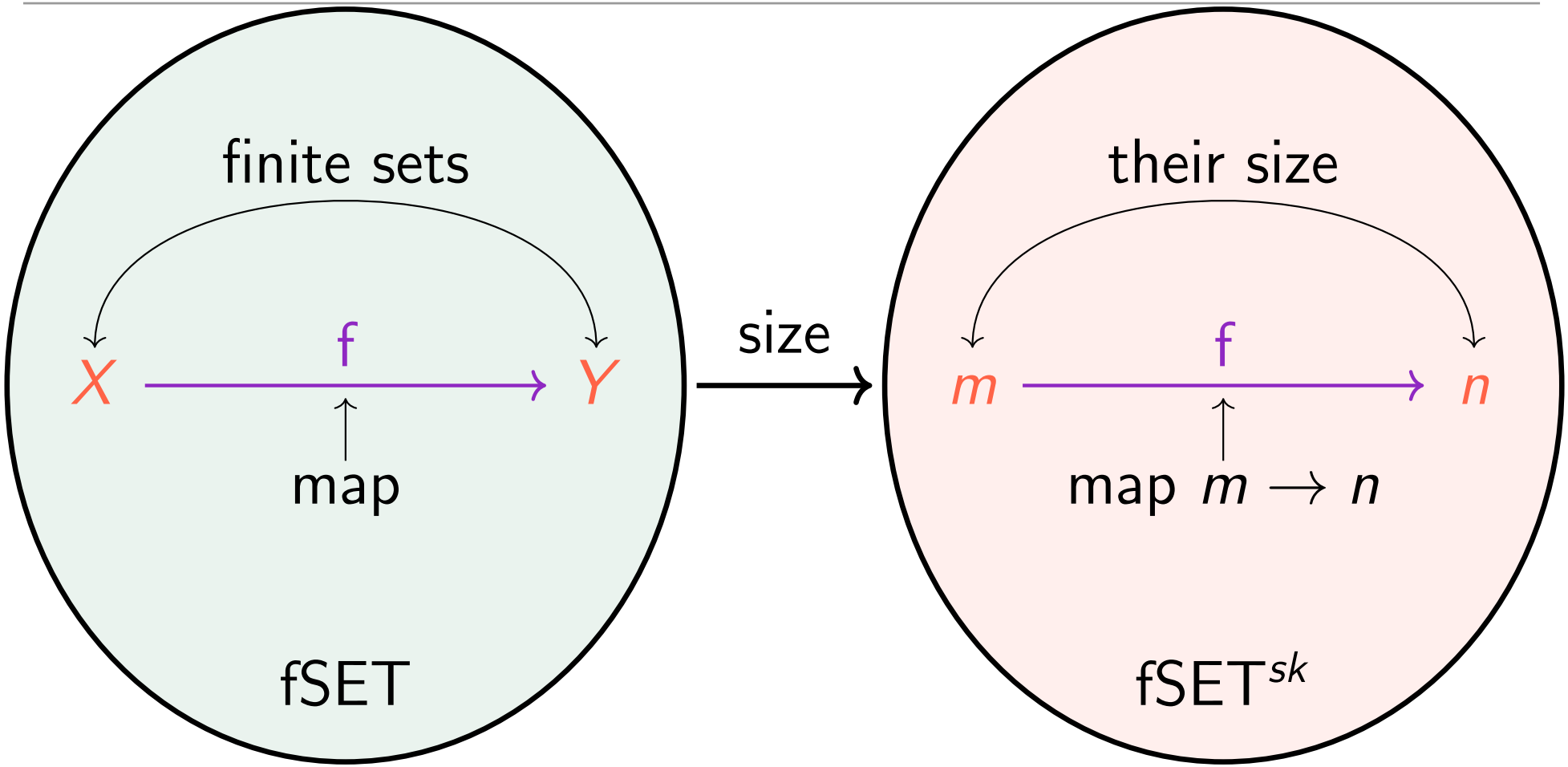
$+$		
		
		

- $\mathbb{Z}/2\mathbb{Z}$ and the fruit group FG are not the same since their sets differ
- They are equivalent=isomorphic "Same up to renaming"

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} FG, \quad 0 \mapsto \text{  }, 1 \mapsto \text{  }$$

- Main point As soon as one varies the underlying set $\cong$ is the true $=$

Category theory goes one step further



- fSET Category of finite sets , $fSET^{sk}$ Objects $\mathbb{N}$, arrows $\hom_{fSET}(m = \{0, ..., m-1\}, n = \{0, ..., n-1\})$
- Equivalence given by the size functor

set-based mathematics: fSET has "more" objects, category theory: Who cares?

For completeness: A formal definition

An isomorphism $F : C \to D$ is a functor such that

$$\exists G : D \to C \text{ with } GF = id_C \text{ and } FG = id_D$$

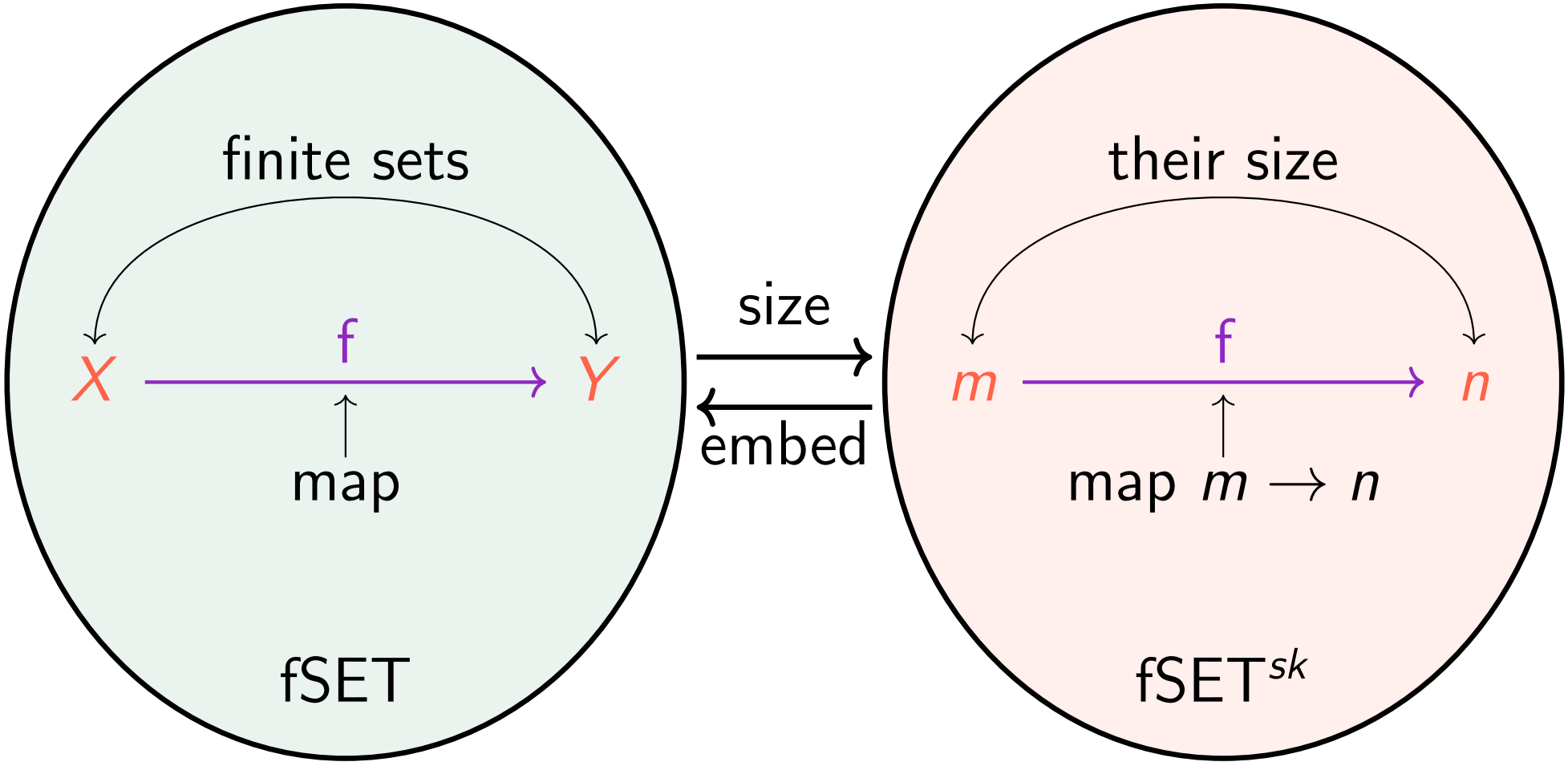
- In this case, C and D are called isomorphic $C \cong D$
- Isomorphisms are bijections on objects and arrows
- This notation coincides with isomorphisms in CAT

An equivalence $F : C \to D$ is a functor such that

$$\exists G : D \to C \text{ with } GF \cong id_C \text{ and } FG \cong id_D \quad (\cong \text{ means natural iso})$$

- In this case, C and D are called equivalent $C \simeq D$
- Equivalences are bijections on arrows
- F is an equivalence $\Leftrightarrow$ F is fully faithful and essentially surjective
- Essentially surjective = All $Y \in D$ are isomorphic to some $F(X)$
- This is the "correct" notion of equal in CAT

Category theory takes itself serious, again



fSET $\not\simeq$ fSET^{sk} but fSET $\simeq$ fSET^{sk} via size and embed

Slogan

A property is preserved by $\simeq$ if and only if it does not involve equations of objects

Thank you for your attention!

I hope that was of some help.