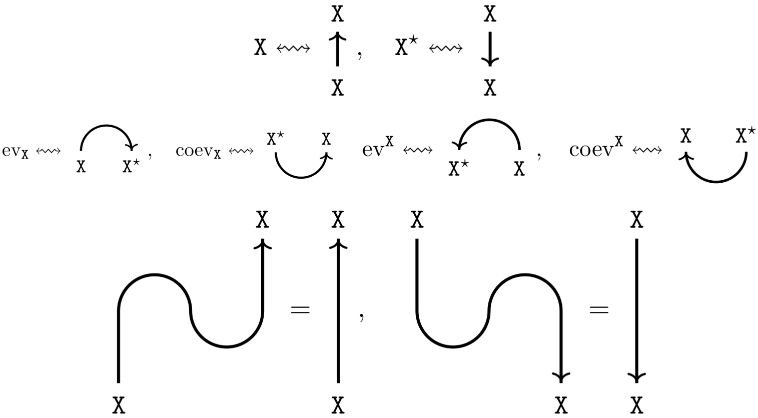
What is...duality in categories?

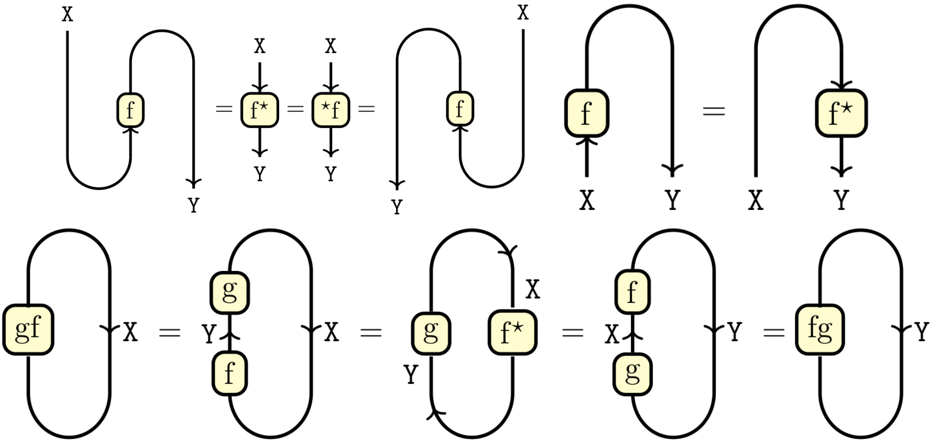
Or: Rigid and pivotal categories

String diagram wish list



- We want dual objects encoded by orientations
- We want (co)evaluations encoded by cups/caps
- We want the zigzag relation

String diagram consequences



- We get dual arrows as above
- We get topological rules how to manipulate these diagrams
- We get diagrammatic proofs of nontrivial facts

Vector spaces again

$$ev\colon XX^\star \to \mathbb{K}$$

$$(x, y^\star) \mapsto y^\star(x)$$

$$coev\colon \mathbb{K} \to (X^\star)X$$

$$1 \mapsto \sum x^\star \otimes x$$

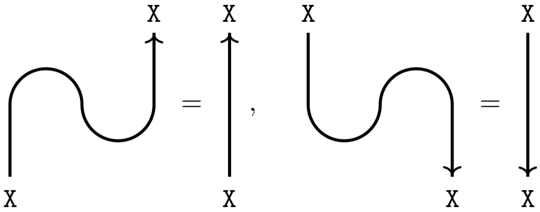
fd$\mathbb{K}$VECT

- In fd$\mathbb{K}$VECT, the dual vector space is the dual object
- In fd$\mathbb{K}$VECT, (co)evaluations are the usual (co)evaluations
- Seems like we need monoidal categories to make this work

For completeness: A formal definition

An object X in a monoidal category C has a dual $X^\star$ if:

- There exists (co)evaluation arrows (four variants)
- They satisfy the zigzag relations



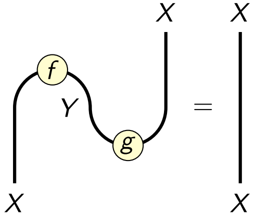
A category is (strictly) pivotal if all objects have duals

- Duals, if they exist, are unique up to unique isomorphism
- rigid is a weaker notion where one distinguishes left/right $^\star X/X^\star$ duals
- Example The category $\mathrm{End}(C)$ is rigid and left/right dual = left/right adjoint

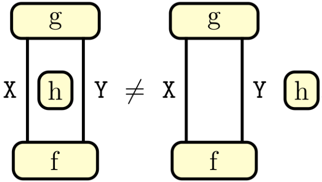
Finally, a planar calculus

Theorem. Two diagrams are equivalent if they are related by scaling or by a planar isotopy

works for f, g being (co)evaluations :



- This is a planar calculus
- Warning This really is planar and not allowed is *e.g.*



Thank you for your attention!

I hope that was of some help.