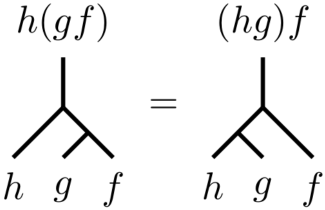
What is...strictification?

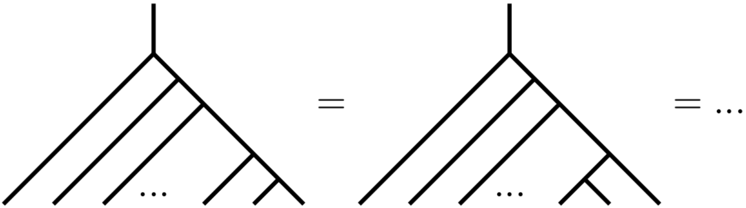
Or: MacLane's coherence theorem

A "wrong" and a "correct" definition

"Wrong" :

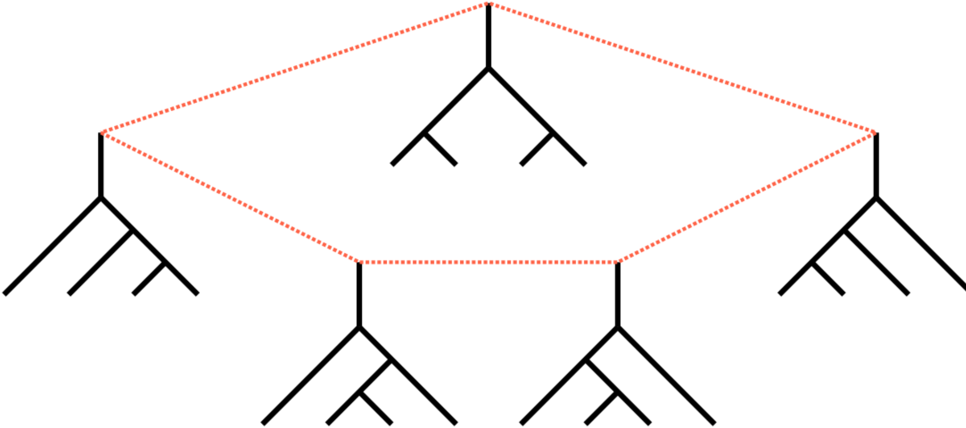


"Correct" :



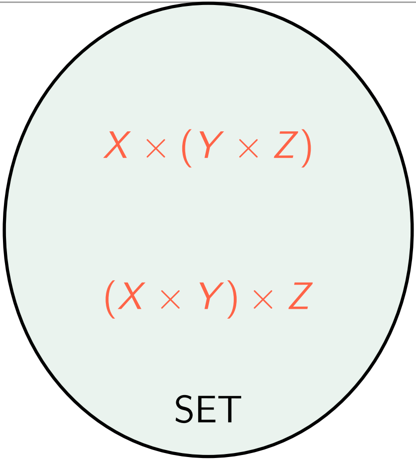
- (A) $h \cdot (g \cdot f) = (h \cdot g) \cdot f$
- (B) Same result regardless of how valid pairs of parentheses are inserted
- "Philosophically correct" Use (B) as the definition and show that (A) $\Leftrightarrow$ (B)

Strategical interlude



- Define a monoid/group/... using $h \cdot (g \cdot f) = (h \cdot g) \cdot f$
- Show that $h \cdot (g \cdot f) = (h \cdot g) \cdot f$ implies all bracketings Coherence theorem
- Forget parenthesis altogether Strictification

Why parenthesis in the first place?



- $X \times (Y \times Z) \neq (X \times Y) \times Z$ as sets Set theory is inflexible
- In order to make SET with $\otimes = \times$ monoidal we need parenthesis
- Use an equivalent category and avoid parenthesis Category theory is flexible

For completeness: A formal definition

A **strict monoidal category** $(C, \otimes, \mathbb{1})$ consists of

- A category C
- A bifunctor $\otimes\colon C \times C \to C$ (write $XY = X \otimes Y$)
- A unit object $\mathbb{1} \in C$

such that

(a) *associativity* holds, *i.e.*

$$X(YZ) = (XY)Z, \quad h \otimes (g \otimes f) = (h \otimes g) \otimes f$$

(b) the *identity law* holds, *i.e.*

$$\mathbb{1}X = X = X\mathbb{1}, \quad \mathrm{id} \otimes f = f = f \otimes \mathrm{id}$$

Theorem (strictification)

Every monoidal category is monoidally equivalent to a strict monoidal category

Some examples

Name	$\otimes$	Strict?	Strictification
SET	$\times$	No	S(SET)
CAT	$\times$	No	S(CAT)
1COB	Juxtaposition	Yes	1COB
nCOB	Juxtaposition	Yes	nCOB
$\mathbb{K}$VECT	$\otimes$	No	S($\mathbb{K}$VECT)
$\mathbb{K}$VECT	$\oplus$	No	S($\mathbb{K}$VECT)
END(C)	$\circ$	Yes	END(C)

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=70, "F"] \arrow[r, "G"] \arrow[r, bend right=70, "H"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & \mathcal{A}' \arrow[r, bend left=70, "F'"] \arrow[r, "G'"] \arrow[r, bend right=70, "H'"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta'"] & \mathcal{A}''
\end{tikzcd}$$

- Often the skeleton $S(C)$ is the strictification but not always
- In general the strictification is an endofunctor category

Thank you for your attention!

I hope that was of some help.