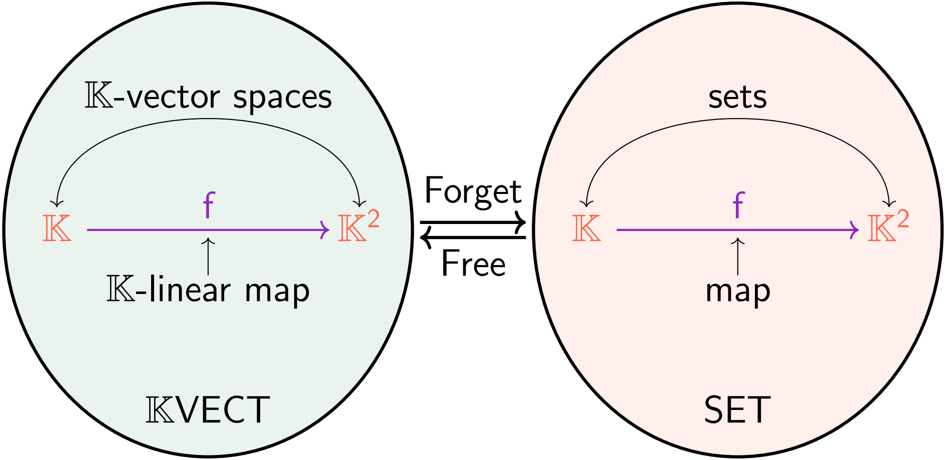
What is...Beck's monadicity theorem?

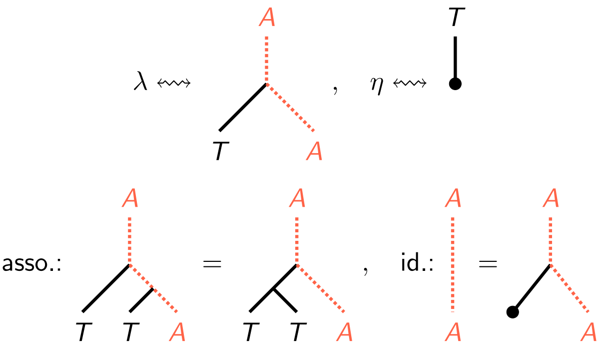
Or: Categories via actions

The (*Free*, *Forget*) adjunction again



- T =Forget∘Free gives a monad on SET
- $T(A) \cong \mathbb{K}\{A\}$ (formal $\mathbb{K}$-linear combinations)
- Question What are T-algebras (a.k.a. T-modules)?

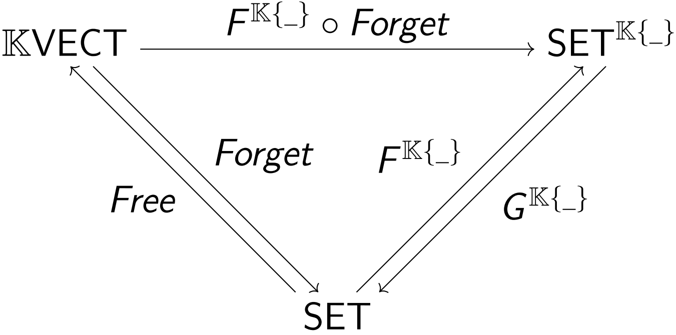
$\mathbb{K}$VECT as an Eilenberg–Moore category?



- ($\lambda\colon \mathbb{K}\{A\} \to A$ + asso. + id.) implies $\mathbb{K}$-linearity on A
- Being a $\mathbb{K}\{\_\}$-intertwiner implies $\mathbb{K}$-linearity of $f\colon A \to B$
- $\mathbb{K}$-vector spaces are $\mathbb{K}\{A\}$-algebras?

$$\mathbb{K}\text{VECT} \simeq \text{SET}^{\mathbb{K}\{\_\}}?$$

$\mathbb{K}$VECT as an Eilenberg–Moore category!



- $F^{\mathbb{K}\{\_\}} \circ Forget$ is an equivalence of categories
- Thus, $\mathbb{K}$VECT is a module category of a monad
- Beck's monadicity theorem characterizes the situations where this works

For completeness: A formal definition/statement

(F, G) adjunction between C, D, and $T = G \circ F$ the associated monad on C

- $\exists!$ comparison functor $K\colon D \to C^T$
- The adjunction is monadic if $K\colon D \xrightarrow{\simeq} C^T$

$G\colon D \to C$ is monadic if $\exists$ left adjoint $F\colon C \to D$ such that

$$K\colon D \xrightarrow{\simeq} C^{G \circ F}$$

Beck's monadicity theorem $G\colon D \to C$ is monadic if and only if

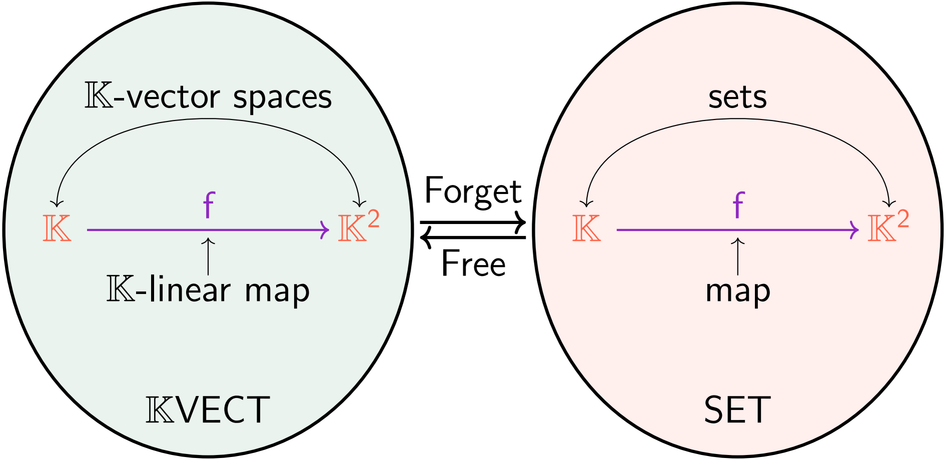
(a) $\exists$ left adjoint $F\colon C \to D$

(b) (If $G(f)$ is an isomorphism, then so is f) is satisfied

(c) D has coequalizers of G-split parallel pairs, and G preserves these

G-split parallel pairs are those parallel arrows in D which G sends to arrows having a split coequalizer

Crude monadicity theorem



(a) $\exists$ left adjoint $F\colon C \to D$

(b) (If $G(f)$ is an isomorphism, then so is f) is satisfied

(c) D has coequalizers and G preserves these

(a), (b) and (c) imply that $G\colon D \to C$ is monadic (Checks for the above)

Thank you for your attention!

I hope that was of some help.