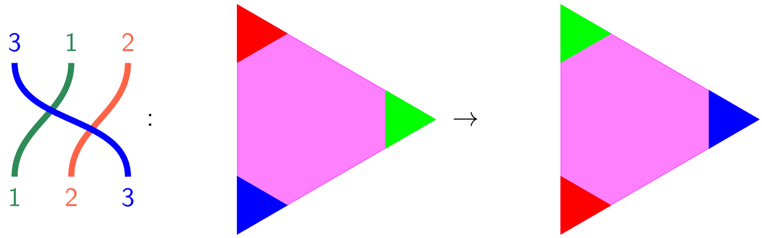
What are...algebras?

Or: Actions in action

Actions and modules

The symmetric group on $\{1,2,3\}$ acts on a triangle via the rule "green=1, red=2, blue=3, and then permute":



- Monoid actions give modules
- Monads ⇜⇝ monoids in functors
- Monad actions give algebras (not sure where the name comes from)

Actions in diagrams

Act. : $\lambda \colon M \times V \to V$

Asso. :

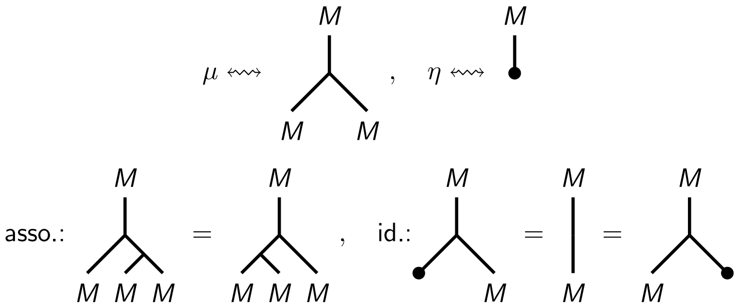
$$\begin{array}{ccc} M \times M \times V & \xrightarrow{id \times \lambda} & M \times V \\ {\scriptstyle \mu \times id}\downarrow & & \downarrow{\scriptstyle \lambda} \\ M \times V & \xrightarrow[\lambda]{} & V \end{array}$$

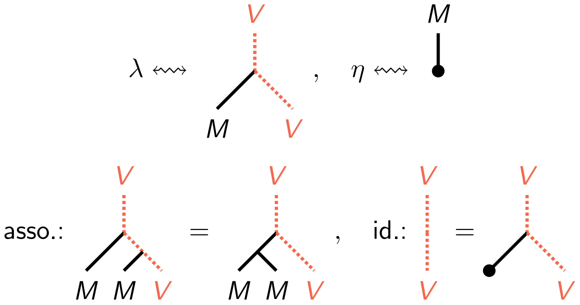
Id. :

$$\begin{array}{ccc} V & \xrightarrow{v \mapsto (1, v)} & M \times V \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle \lambda} \\ & & V \end{array}$$

- Above The definition of a monoid action using commutative diagrams
- Let us formulate this in a category!

Monoids and actions in string-type diagrams





For completeness: A formal definition

A $T = (T, \mu, \eta)$-algebra is a tuple (A, λ) consisting of:

- an object A of C
- nat trafo $\lambda \colon TA \Rightarrow A$

satisfying associativity and identity as below (the diagrams should commute):

Asso. :

$$\begin{array}{ccc} TTA & \xrightarrow{id \times \lambda} & TA \\ {\scriptstyle \mu \times id}\downarrow & & \downarrow{\scriptstyle \lambda} \\ TA & \xrightarrow[\lambda]{} & A \end{array}$$

Id. :

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \lambda} \\ & & A \end{array}$$

Eilenberg–Moore – category of algebras/modules

Eilenberg–Moore category C^T

Objects are T-algebras

Arrows are intertwiners

- $\exists$ Eilenberg–Moore category C^T associated to T Category of modules
- There is a free-forget adjunction between C^T and C
- The monad T arises via this adjunction

Thank you for your attention!

I hope that was of some help.