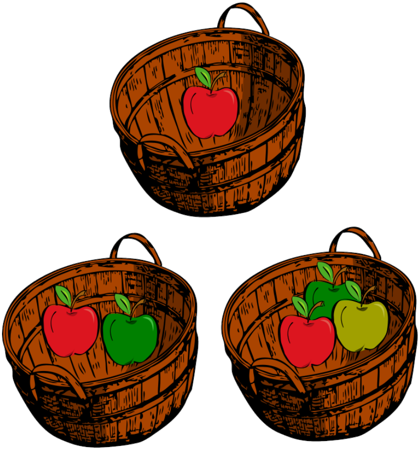
What are...monads?

Or: The natural numbers as functors

Counting is categories!?



- $\mathbb{N}_0 = \{0, 1, 2, ...\}$ forms arguably the most important monoid set-based
- Question What is the categorical analog?
- Monads are not quite the answer, but come close

Monoids revisited

Group-like structures					
	Totality$^{\alpha}$	Associativity	Identity	Invertibility	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Unital magma	Required	Unneeded	Required	Unneeded	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Inverse semigroup	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Commutative monoid	Required	Required	Required	Unneeded	Required
Group	Required	Required	Required	Required	Unneeded
Abelian group	Required	Required	Required	Required	Required

- A monoid is a set Do not really like that...
- A monoid has an operation That's ok
- The operation satisfies associativity and unitality Also ok

This already looks categorical

Mult. : $\mu\colon M \times M \to M$

Asso. :

$$\begin{CD} M \times M \times M @>{id \times \mu}>> M \times M \\ @V{\mu \times id}VV @VV{\mu}V \\ M \times M @>>{\mu}> M \end{CD}$$

Id. :

$$\begin{array}{ccc} M & \xrightarrow{a \mapsto (1,a)} & M \times M \\ {\scriptstyle a \mapsto (a,1)}\downarrow & \overset{=}{\searrow} & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow[\mu]{} & M \end{array}$$

- Above The definition of a monoid using commutative diagrams
- Let us formulate this in a category!

For completeness: A formal definition

A monad (T, μ, η) is a triple consisting of:

- an endofunctor $T\colon C \to C$
- two nat trafos $\mu\colon TT \Rightarrow T$, $\eta\colon id_C \to T$

satisfying associativity and identity as below (the diagrams should commute):

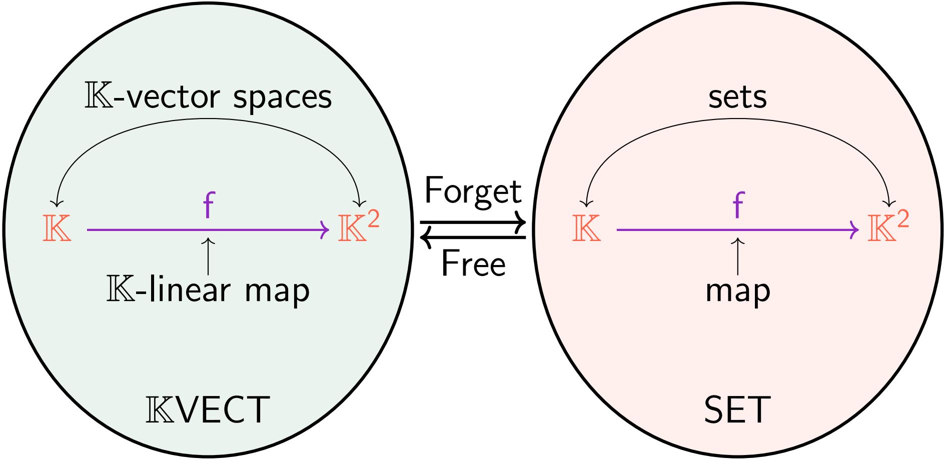
Asso. :

$$\begin{array}{ccc} TTT & \xrightarrow{id \times \mu} & TT \\ {\scriptstyle \mu \times id}\downarrow & & \downarrow{\scriptstyle \mu} \\ TT & \xrightarrow[\mu]{} & T \end{array}$$

Id. :

$$\begin{array}{ccc} T & \xrightarrow{id\eta} & TT \\ {\scriptstyle \eta id}\downarrow & \searrow^{=} & \downarrow{\scriptstyle \mu} \\ TT & \xrightarrow[\mu]{} & T \end{array}$$

Monads and adjunctions



- T =Forget∘Free gives a monad on SET
- General Every adjoint pair gives a monad
- General Every monad arises from an adjoint pair

Thank you for your attention!

I hope that was of some help.