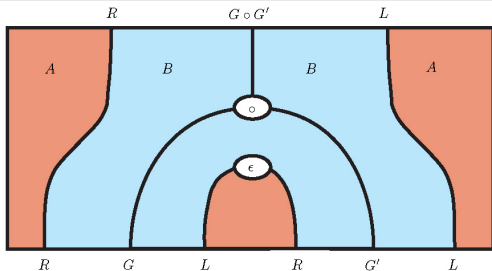


What are...adjoint functors diagrammatically?

Or: Zigzag!

Two-dimensional algebra

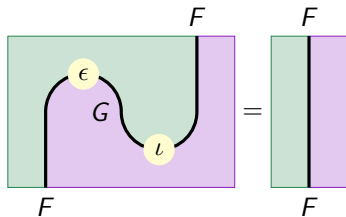


- ▶ Functor calculus gets a 2d flavor via string diagrams
- ▶ Important Do not draw identity functors id_C , e.g.

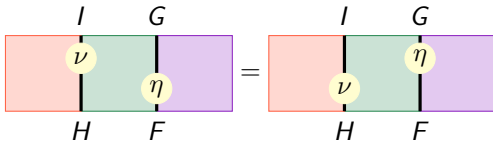
$$\begin{array}{l}
 F: C \rightarrow D, G: D \rightarrow C \\
 \epsilon: FG \Rightarrow id_D
 \end{array}
 \quad \Leftarrow \quad
 \begin{array}{c}
 \text{Diagram 1} \\
 \text{A purple semi-circle with a yellow circle labeled } \epsilon \text{ above it, sitting on a green rectangular background. The left and right ends of the semi-circle are labeled } F \text{ and } G \text{ respectively.}
 \end{array}
 =
 \begin{array}{c}
 \text{Diagram 2} \\
 \text{A purple semi-circle with a yellow circle labeled } \epsilon \text{ above it, sitting on a green rectangular background. A vertical line labeled } id_D \text{ extends upwards from the top of the semi-circle. The left and right ends of the semi-circle are labeled } F \text{ and } G \text{ respectively.}
 \end{array}$$

- ▶ Goal Rediscover adjoint functors using plane geometry only

Zigzag



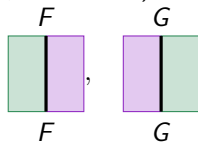
- ▶ There are several compatibility conditions we need anyway, e.g.



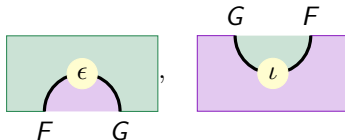
- ▶ The zigzag relation is a genuine and crucial relation planar diagrams satisfy
- ▶ Use this to define certain functors

What do we need for the zigzag?

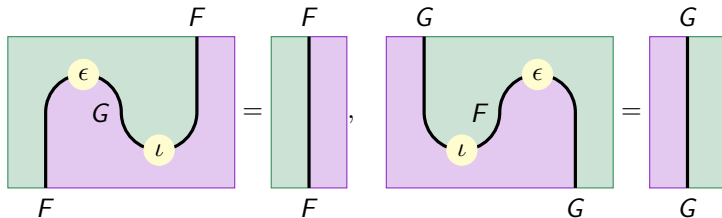
- ▶ We need functors ($F: C \rightarrow D, G: D \rightarrow C$) in opposite directions



- ▶ We need nat trafos $\epsilon: FG \Rightarrow id_D$ and $\iota: id_C \Rightarrow GF$



- ▶ We need the relations



For completeness: A formal definition

Two functors $(F, G) = (F: C \rightarrow D, G: D \rightarrow C)$ for an adjoint pair if:

- ▶ There exists a counit $\epsilon: FG \Rightarrow id_D$
- ▶ There exists a unit $\iota: id_C \Rightarrow GF$
- ▶ They satisfy the zigzag relations:

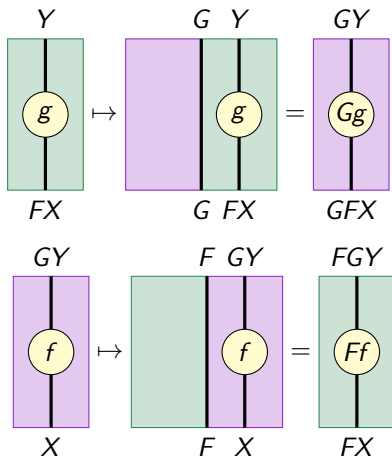
$$\begin{array}{ccc} F & \xrightarrow{id_F \otimes \iota} & FGF \xrightarrow{\epsilon \otimes id_F} F \\ & \searrow & \nearrow \\ & & id_F \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{\iota \otimes id_G} & GFG \xrightarrow{id_G \otimes \epsilon} G \\ & \searrow & \nearrow \\ & & id_G \end{array}$$

In this case F is the left adjoint of G , and G is the right adjoint of F

- ▶ A functor might not have left/right adjoints
- ▶ If they exist, then they are unique up to unique isomorphism

The hom adjunction



- ▶ For (F, G) we have $\text{hom}_{\mathcal{D}}(FX, Y) \cong \text{hom}_{\mathcal{C}}(X, GY)$
- ▶ How can we see this? Use the diagrams above!

Thank you for your attention!

I hope that was of some help.