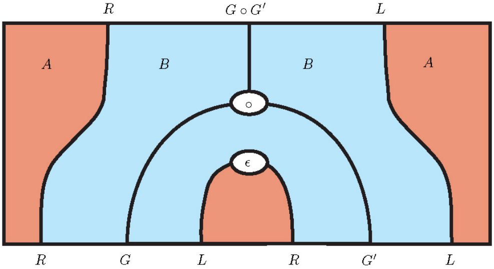
What are...adjoint functors diagrammatically?

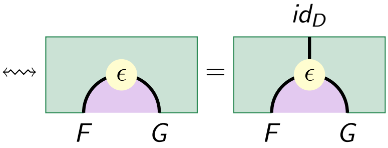
Or: Zigzag!

Two-dimensional algebra



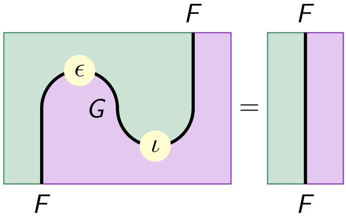
- Functor calculus gets a 2d flavor via string diagrams
- Important Do not draw identity functors id_C, *e.g.*

$$F\colon C \to D, G\colon D \to C$$
$$\epsilon\colon FG \Rightarrow id_D$$

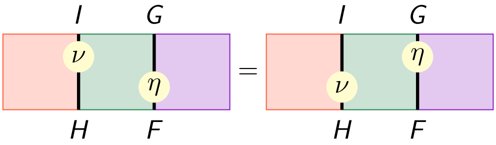


- Goal Rediscover adjoint functors using plane geometry only

Zigzag



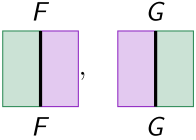
- There are several compatibility conditions we need anyway, *e.g.*



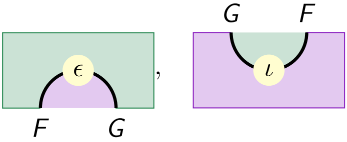
- The zigzag relation is a genuine and crucial relation planar diagrams satisfy
- Use this to define certain functors

What do we need for the zigzag?

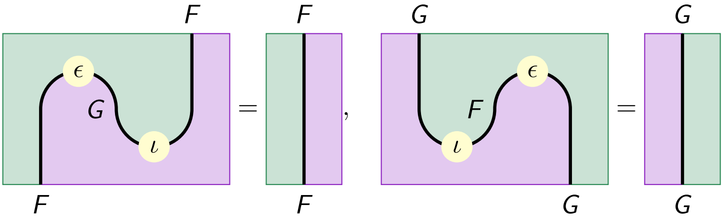
- We need functors ($F\colon C \to D, G\colon D \to C$) in opposite directions



- We need nat trafos $\epsilon\colon FG \Rightarrow id_D$ and $\iota\colon id_C \Rightarrow GF$



- We need the relations



For completeness: A formal definition

Two functors $(F, G) = (F: C \to D, G: D \to C)$ for an adjoint pair if:

- There exists a counit $\epsilon: FG \Rightarrow id_D$
- There exists a unit $\iota: id_C \Rightarrow GF$
- They satisfy the zigzag relations:

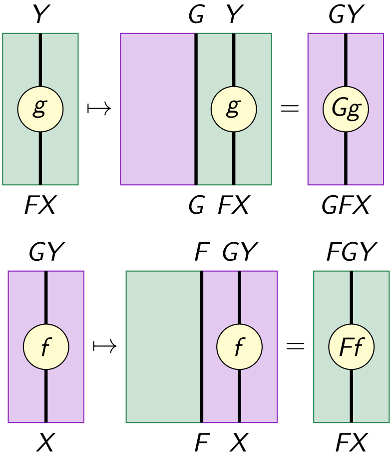
$$\underbrace{F \xrightarrow{id_F \otimes \iota} FGF \xrightarrow{\epsilon \otimes id_F} F}_{id_F}$$

$$\underbrace{G \xrightarrow{\iota \otimes id_G} GFG \xrightarrow{id_G \otimes \epsilon} G}_{id_G}$$

In this case F is the left adjoint of G, and G is the right adjoint of F

- A functor might not have left/right adjoints
- If they exist, then they are unique up to unique isomorphism

The hom adjunction



- For (F, G) we have $\hom_D(FX, Y) \cong \hom_C(X, GY)$
- How can we see this? Use the diagrams above!

Thank you for your attention!

I hope that was of some help.