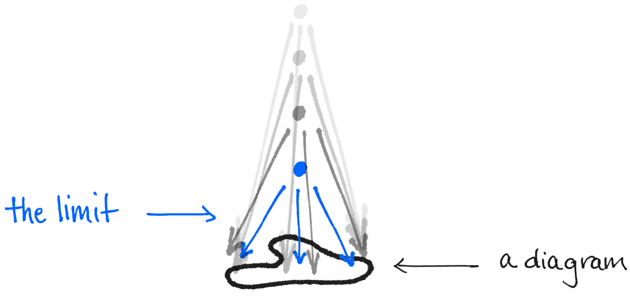
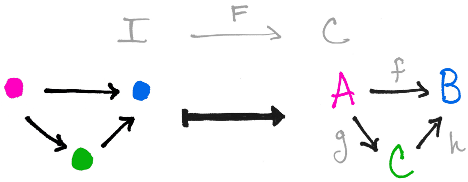
What are...limits?

Or: Universal diagrams

A minimal type of diagram



- Given a diagram F such as



- The limit should be a universal object/arrow minimally associated to F

Products

$$\begin{tikzcd}
 & & C \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & & \\
A_1 & A_2 & A_3 & A_4 & \cdots
\end{tikzcd}$$

There are no morphisms between the A_i.

Hence "discrete".

The **product** (limit) of the data in red is the data in blue.

= the diagram you start with

= the limit of that diagram

- Given the following diagram

$$\bullet \qquad \bullet \qquad \bullet \qquad \bullet \qquad \ldots$$

- The associated limit is the product

Pullback

$$
\begin{tikzcd}
A \times_C B \arrow[r, "\text{project}"] \arrow[d, "\text{project}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}
$$

"a pullback diagram"

The **pullback** (limit) of the data in red <u>is</u> the data in blue.

People use this little symbol to say "Hey! Not only does $A \times_C B$ fit into this diagram, it does so *universally*."

= the diagram you start with

= the limit of that diagram

- Given the following diagram

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

- The associated limit is the pullback

For completeness: A formal definition

A pair (L, ϕ) associated to a diagram $\mathcal{D}\colon J \to C$ is...

- ...a limit if the universal property given by the left diagram below holds
- ...a colimit if the universal property given by the right diagram below holds

$$
\begin{tikzcd}
& N \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\
& L \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\
F(X) \arrow[rr, "F(f)"'] & & F(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[rr, "F(f)"] \arrow[dr, "\phi_X"'] \arrow[ddr, bend right, "\psi_X"'] & & F(Y) \arrow[dl, "\phi_Y"] \arrow[ddl, bend left, "\psi_Y"] \\
& L \arrow[d, dashed, "u"] & \\
& N &
\end{tikzcd}
$$

- These might not exists
- If they exist, then they are unique up to unique isomorphism
- The notions limit and colimit are dual

Limits everywhere

This diagram	is this functor, and	its limit is called	its colimit is called
empty	$\mapsto$	the terminal object	the initial object
$A \quad B \quad C$	$\bullet \quad \bullet \quad \bullet \;\mapsto\; A \quad B \quad C$	the product	the coproduct
$A \to C \leftarrow B$	$\bullet \to \bullet \leftarrow \bullet \;\mapsto\; A \to C \leftarrow B$	the pullback	—
$A \leftarrow C \to B$	$\bullet \leftarrow \bullet \to \bullet \;\mapsto\; A \leftarrow C \to B$	—	the pushout
$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots$	$\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots \;\mapsto\; A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots$	the inverse limit	—
$A_1 \to A_2 \to A_3 \to \cdots$	$\bullet \to \bullet \to \bullet \to \cdots \;\mapsto\; A_1 \to A_2 \to A_3 \to \cdots$	—	the direct limit
$A \rightrightarrows B$	$\bullet \rightrightarrows \bullet \;\mapsto\; A \rightrightarrows B$	the equalizer	the coequalizer

A lot of familiar concepts are obtained in this way!

Thank you for your attention!

I hope that was of some help.