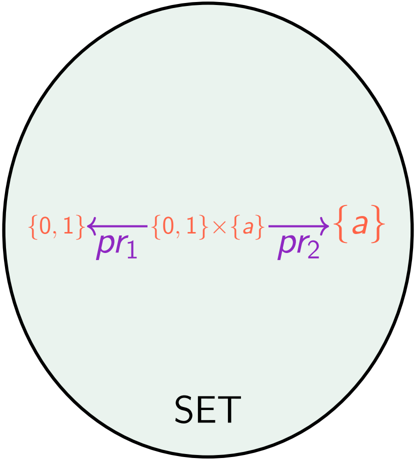
What are...universal properties?

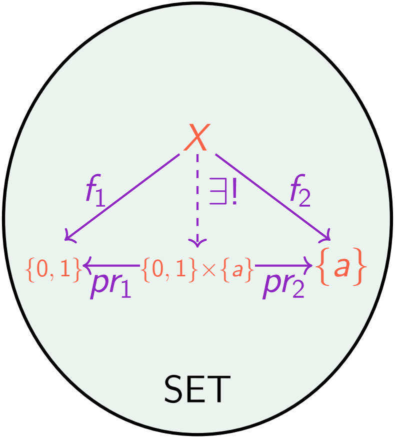
Or: Abstract nonsense in perfection

Tuples



- Products in SET are tuples, *e.g.* $\{0,1\}\times\{a\} = \{(0,a),(1,a)\}$
- Tuples have two independent components
- Question What does this imply?

Uniqueness



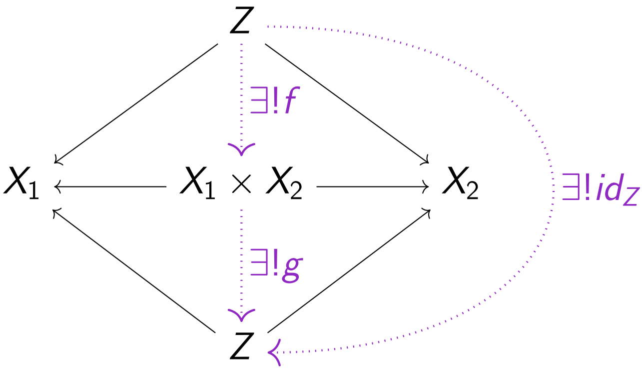
- Say we have $X = \{0, 1, 2\}$ and

$$f_1 : \{0, 1, 2\} \to \{0, 1\}, 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1, \quad f_2 : \{0, 1, 2\} \to \{a\}, i \mapsto a$$

- There exists a unique map such that the above diagram commutes:

$$f : \{0, 1, 2\} \to \{(0, a), (1, a)\}, 0 \mapsto (0, a), 1 \mapsto (1, a), 2 \mapsto (1, a)$$

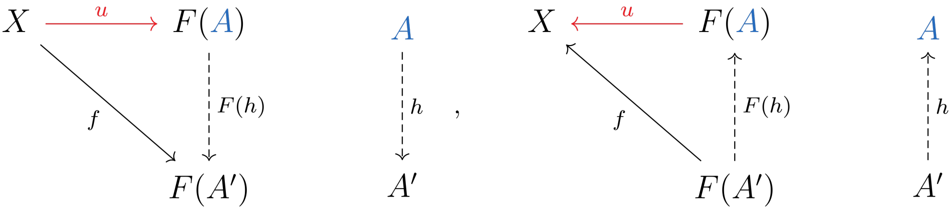
Products are very unique



- Assume there is another product Z
- Play the glue-universal-diagrams together trick
- It follows that $Z \cong X_1 \times X_2$ uniquely

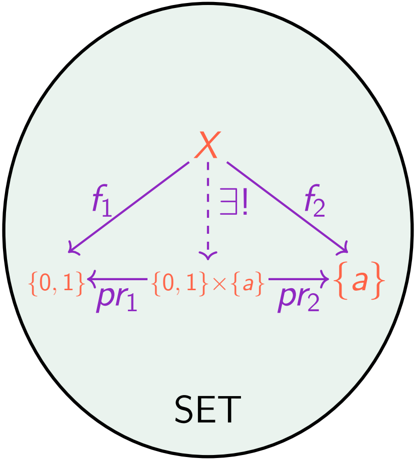
For completeness: A formal definition

- $F\colon C \to D$, $X \in D$, a universal morphism from X to F is a pair $\big(A, u\colon X \to F(X)\big)$ such that $\exists! h$ making the left diagram below commute
- $F\colon C \to D$, $X \in D$, a universal morphism from F to X is a pair $\big(A, u\colon F(X) \to X\big)$ such that $\exists! h$ making the right diagram below commute



- These might not exists
- If they exist, then they are unique up to unique isomorphism

Back to products



- Take $\Delta\colon \mathrm{SET} \to \mathrm{SET} \times \mathrm{SET}$
- $\big(X \times Y, (pr_1, pr_2)\big)$ is a universal morphism from Δ to (X, Y)
- $C \leftrightsquigarrow$ category of interest $D \leftrightsquigarrow$ indexing category

Thank you for your attention!

I hope that was of some help.