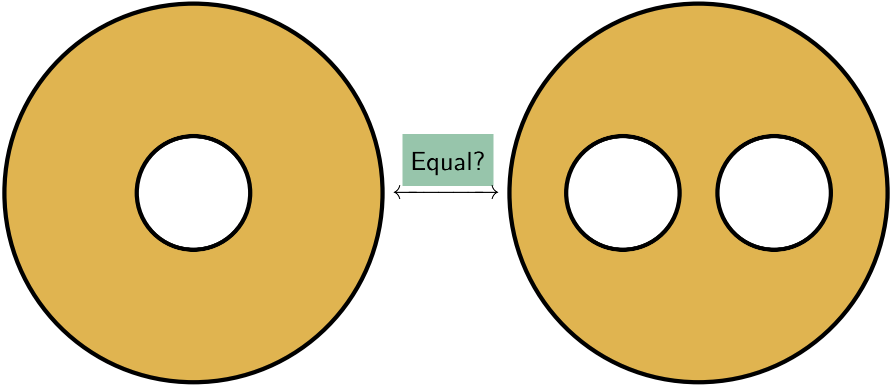
What is...the fundamental group?

Or: How not to hang pictures on walls

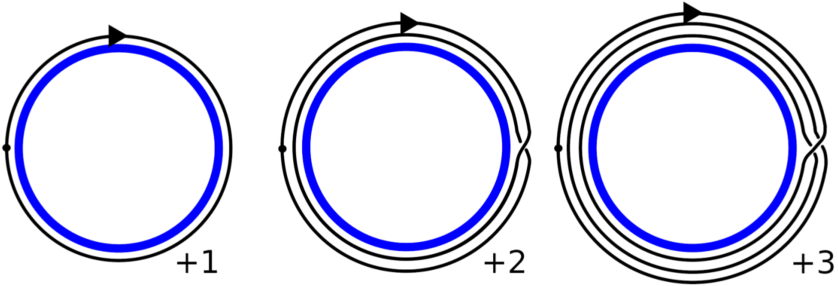
We need an invariant



Are these two homotopic? Obviously not! But how can we prove this?

Key. Find an algebraic invariant that we can compute

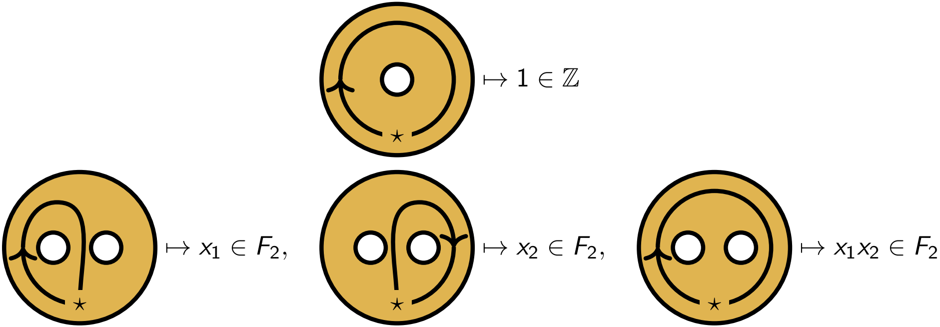
Winding around the circle S^1



$\leftrightsquigarrow \mathbb{Z}$

- Winding around a circle has a group structure – the one of $\mathbb{Z}$
- Winding clockwise once $\leftrightsquigarrow +1 \in \mathbb{Z}$
- Winding anticlockwise once $\leftrightsquigarrow -1 \in \mathbb{Z}$
- The group $\mathbb{Z}$ is a homotopy invariant of S^1

Two different groups ⇒ two different spaces



- The group π_1 associated to the disc with a hole is $\mathbb{Z}$
- The group π_1 associated to the disc with two holes is F_2 (free group in two generators)
- These groups are not isomorphic ⇒ the spaces are not homotopic Invariance

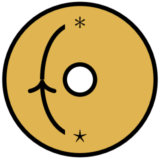
For completeness: A formal definition

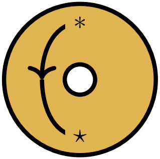
For a topological space X take loops $\gamma\colon [0,1] \to X$ based at $\star \in X$

(a) Let $\pi_1(X,\star)$ be the set of equivalence classes of loops based at $\star$ modulo homotopy

(b) $\pi_1(X,\star)$ has a group structure given by concatenation

- Slight catch. This is only a group structure by using homotopy
- For path connected X we have $\pi_1(X,\star) \cong \pi_1(X,*)$ Write $\pi_1(X)$

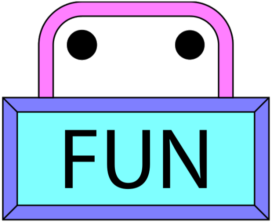


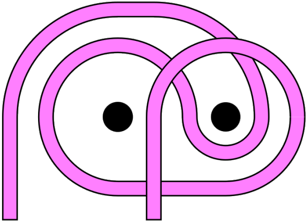
$\circ$ path in $\pi_1(X,\star)$ $\circ$


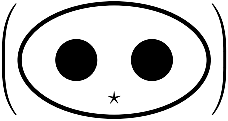
$=$ path in $\pi_1(X,*)$

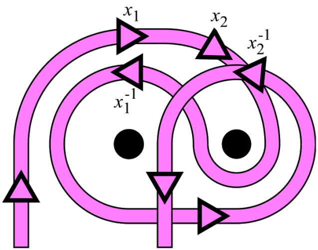
- $(X \simeq Y) \Rightarrow (\pi_1(X) \cong \pi_1(Y))$ Invariance

Using two nails works, using one fails





π_1  = free group in two generators x_1, x_2



$\rightsquigarrow$

$$x_1x_2x_1^{-1}x_2^{-1} \in \pi_1 \xrightarrow[\text{remove left nail}]{x_1\to 1} x_2x_2^{-1} = 1$$

The picture falls down

$$x_1x_2x_1^{-1}x_2^{-1} \in \pi_1 \xrightarrow[\text{remove right nail}]{x_2\to 1} x_1x_1^{-1} = 1$$

The picture falls down

Thank you for your attention!

I hope that was of some help.