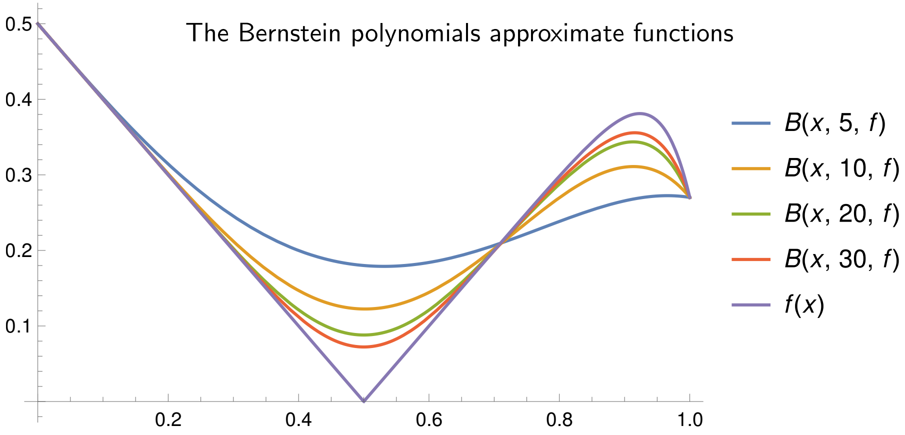
What is...cellular approximation of spaces?

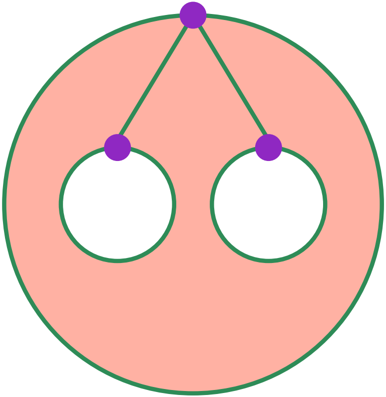
Or: My polynomials have cells

A classic – the Stone–Weierstrass theorem



- Every continuous function can be nicely approximated by polynomials
- IMHO, this very surprising:
  (a) Continuous functions are sometimes really badly behaved
  (b) Polynomials are very well-behaved functions

Cell complexes are like polynomials



Cell complexes...

...are constructed from easy "variables" X

...have a "degree" X^n

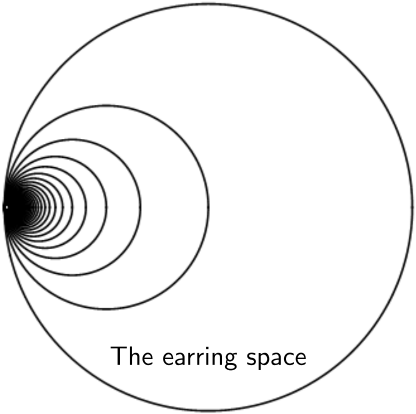
...are closed under "addition" $\coprod$

...are closed under "multiplication" $\times$

- Is there a Stone–Weierstrass theorem for topological spaces?
- Topological spaces are sometimes really badly behaved
- Cell complexes are very well-behaved spaces

What to expect?

Not homotopic to a cell complex:



- Not every space is homotopy equivalent $\simeq$ to a cell complex
- Whitehead's theorem: "$\simeq = \simeq_{weak}$" for nice spaces
- Maybe we can work with weak homotopy equivalence $\simeq_{weak}$?

For completeness: A formal statement

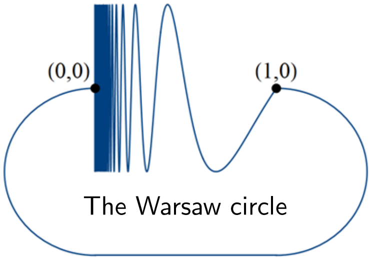
Every space X is weakly homotopy equivalent to a cell complex Y

CW approximation

- Weak homotopy equivalence $f\colon A \xrightarrow{\simeq_{weak}} B$ is a map inducing isomorphisms

$$f_*\colon \pi_*(A) \xrightarrow{\cong} \pi_*(B)$$

- Precisely, the above means there exists Y and $f\colon Y \xrightarrow{\simeq_{weak}} X$
- $\simeq_{weak}$ is strictly weaker than $\simeq$, *e.g.*

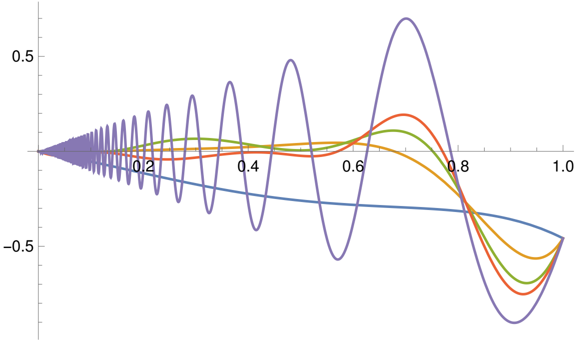


$\simeq_{weak}$ point

$\not\simeq$ point

Some flaws along the way

The Bernstein polynomials are sometimes "bad approximations"



- Stone–Weierstrass is not perfect: some functions are hard to approximate
- Some spaces have "weird" approximate cell complexes, *e.g.*
  (a) The Warsaw circle is $\simeq_{weak}$ to a point, but certainly is not a point
  (b) The earring space H is compact, but

$$\pi_{\neq 1}(H) \cong 0, \quad \pi_1(H) \cong \text{huge+complicated}$$

and its cell approximation is not finite (because of $\pi_1(H)$)

Thank you for your attention!

I hope that was of some help.