What is...the universal coefficient theorem?

Or: Working integrally rocks

Universal coefficient theorem (UCT): Mind your numbers!

$$\dots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \underline{\mathbb{Z}} \Rightarrow H_n \cong \begin{cases} \mathbb{Z} \text{ if } n = 0 \\ \mathbb{Z}/2\mathbb{Z} \text{ if } n \text{ is odd} \\ 0 \text{ else} \end{cases}$$

$$\dots \xrightarrow{0} \mathbb{Q} \xrightarrow{\cdot 2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{\cdot 2} \mathbb{Q} \xrightarrow{0} \underline{\mathbb{Q}} \Rightarrow H_n \cong \begin{cases} \mathbb{Q} \text{ if } n = 0 \\ 0 \text{ else} \end{cases}$$

$$\dots \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \underline{\mathbb{Z}/2\mathbb{Z}} \Rightarrow H_n \cong \mathbb{Z}/2\mathbb{Z}$$

Underline $= C_0$

- H_* depends on the input space and the “types of numbers” used
- The UCT explains how different types of numbers are related

Chain complexes with coefficients

$$\dots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \underline{\mathbb{Z}} = C_*(\mathbb{R}P^\infty)$$

$$\dots \xrightarrow{0} \mathbb{Q} \xrightarrow{\cdot 2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{\cdot 2} \mathbb{Q} \xrightarrow{0} \underline{\mathbb{Q}} = C_*(\mathbb{R}P^\infty) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\dots \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \underline{\mathbb{Z}/2\mathbb{Z}} = C_*(\mathbb{R}P^\infty) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

Underline $= C_0$

- $\mathbb{Z} \to R, 1 \mapsto 1$ gives a way to interpret integers as elements of any ring R
- Formally this can be encoded using $\_ \otimes_{\mathbb{Z}} R$
- We get $C_*(X, R) = C_*(X) \otimes_{\mathbb{Z}} R$ chain complexes with coefficients
- Some numbers will become invertible or zero (divisors) in $C_*(X, R)$

Homology with coefficients

$$H_n(\mathbb{R}P^\infty) \cong \begin{cases} \mathbb{Z} \text{ if } n=0 \\ \mathbb{Z}/2\mathbb{Z} \text{ if } n \text{ is odd} \\ 0 \text{ else} \end{cases}$$

$$H_n(\mathbb{R}P^\infty, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} \text{ if } n=0 \\ 0 \text{ else} \end{cases} \cong H_n(\mathbb{R}P^\infty) \otimes_{\mathbb{Z}} \mathbb{Q}$$ Same

$$H_n(\mathbb{R}P^\infty, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_n(\mathbb{R}P^\infty) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \text{ if } n=0 \text{ or } n \text{ odd} \\ 0 \text{ else} \end{cases}$$ Different

- We get $H_*(X,R)$ =homology of $C_*(X,R)$ homology with coefficients
- We could also naively change coefficients $H_*(X) \otimes_{\mathbb{Z}} R$
- The UCT measures the difference between $H_*(X,R)$ and $H_*(X) \otimes_{\mathbb{Z}} R$

For completeness: A formal definition

For any $\mathbb{Z}$-module R singular homology satisfies

- There exists an exact sequence

$$0 \to \big(H_n(X) \otimes_{\mathbb{Z}} R\big) \to H_n(X, R) \to \mathrm{Tor}(H_{n-1}(X), R) \to 0$$

- This sequence splits (not naturally)
- We have a direct sum decomposition

$$H_n(X, R) \cong \big(H_n(X) \otimes_{\mathbb{Z}} R\big) \oplus \mathrm{Tor}(H_{n-1}(X), R)$$

- $\mathbb{Z}$ is hence the "universal" coefficient group
- $\mathrm{Tor}(H_{n-1}(X), R)$ measures how far $\_ \otimes_{\mathbb{Z}} \_$ is from being exact
- There is also a version for cohomology
- This is a statement "in algebra" and holds more generally

Tor in a bit more detail

$\mathrm{Tor}(A, B)$ is the homology of any free resolution of A tensored with B

Tor measures the failure of $\_ \otimes_{\mathbb{Z}} \_$ being exact:

$$0 \to B \to C \to D \to 0 \text{ exact} \Rightarrow$$

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \to A \otimes_{\mathbb{Z}} B \to A \otimes_{\mathbb{Z}} C \to A \otimes_{\mathbb{Z}} D \to 0 \text{ exact}$$

Basic tools for computing Tor:

- $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$ Commutative
- $\mathrm{Tor}(\bigoplus_i A_i, B) \cong \bigoplus_i \mathrm{Tor}(A_i, B)$ Additive
- $\mathrm{Tor}(A, B) \cong 0$ if A or B is torsionfree Often trivial, *e.g.* $\mathrm{Tor}(\mathbb{Q}, B) \cong 0$
- $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, B) \cong \ker(B \xrightarrow{\cdot n} B)$ Torsion, *e.g.* $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$

Thank you for your attention!

I hope that was of some help.