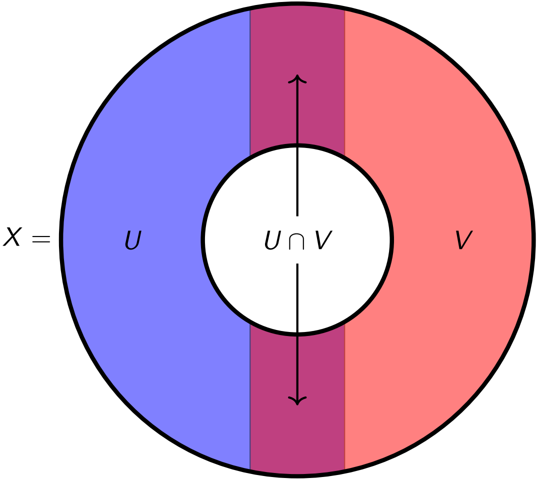
What is...the Mayer–Vietoris sequence?

Or: More than the sum of its parts!?

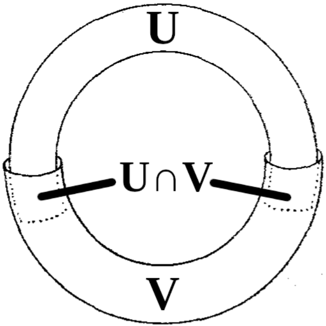
The sum of its parts!



If we know $H_*(U), H_*(V), H_*(U \cap V)$, shouldn't we be able to compute $H_*(X)$?

Cutting into pieces

The sum of its parts?

$X =$ 

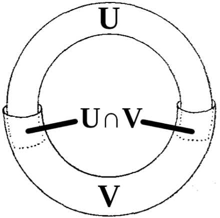
$U, V \leftrightsquigarrow$ cylinder, $U \cap V \leftrightsquigarrow$ two cylinders

- The Hilbert–Poincaré polynomials are

$$P(X) = 1 + 2t + t^2, \quad P(U) = P(V) = 1 + t, \quad P(U \cap V) = 2 + 2t$$

- Question Is there any relation between these?

The sum of its parts! Well, almost...

$X =$ 

- We get inequations on Hilbert–Poincaré polynomials:

$$1 + 2t + t^2 \leq (1+t) + (1+t) + t \cdot (2+2t)$$

$$P(X) \leq P(U) + P(V) + t \cdot P(U \cap V)$$

$$(1+t) + (1+t) \leq (1 + 2t + t^2) + (2+2t)$$

$$P(U) + P(V) \leq P(X) + P(U \cap V)$$

$$t \cdot (2+2t) \leq t \cdot \big((1+t) + (1+t)\big) + (1 + 2t + t^2)$$

$$t \cdot P(U \cap V) \leq t \cdot \big(P(U) + P(V)\big) + P(X)$$

- The Mayer–Vietoris sequences tells you what to do to get equalities

For completeness: A formal statement

For X any topological space with subspaces U, V whose interior cover X, we have an exact sequence

$$\cdots \xrightarrow{\partial_*} H_n(U \cap V) \xrightarrow{(i_*,j_*)} H_n(U) \oplus H_n(V) \xrightarrow{k_*-l_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(U \cap V) \to \cdots$$

$$\cdots \to H_1(X) \xrightarrow{\partial_*} H_0(U \cap V) \xrightarrow{(i_*,j_*)} H_0(U) \oplus H_0(V) \xrightarrow{k_*-l_*} H_0(X) \to 0$$

This needs a choice of order for U, V

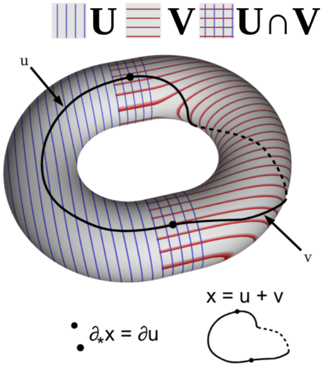
- Here we use the inclusions $i, j\colon U \cap V \hookrightarrow U, V$ and $k, l\colon U, V \hookrightarrow X$
- The boundary map ∂_* (next slide) lowers the degree
- The three ways to cut out three bits give the inequations, *e.g.*

$$H_*(U) \oplus H_*(V) \to H_*(X) \to H_{*-1}(U \cap V)$$

$$P(X) \leq P(U) + P(V) + t \cdot P(U \cap V)$$

- This works *mutatis mutandis* for cohomology as well

Boundaries $\partial_n : H_n(X) \to H_{n-1}(U \cap V)$



- A cycle x in $H_n(X)$ is sum of two chains u, v in U, V
- $\partial_n(x) = \partial_n(u + v) = 0 \Rightarrow \partial_n(u) = -\partial_n(v)$
- Both, $\partial_n(u), \partial_n(v)$ lie in $H_{n-1}(U \cap V)$
- Define $\partial_n(x) = \partial_n(u) \in H_{n-1}(U \cap V)$

Thank you for your attention!

I hope that was of some help.