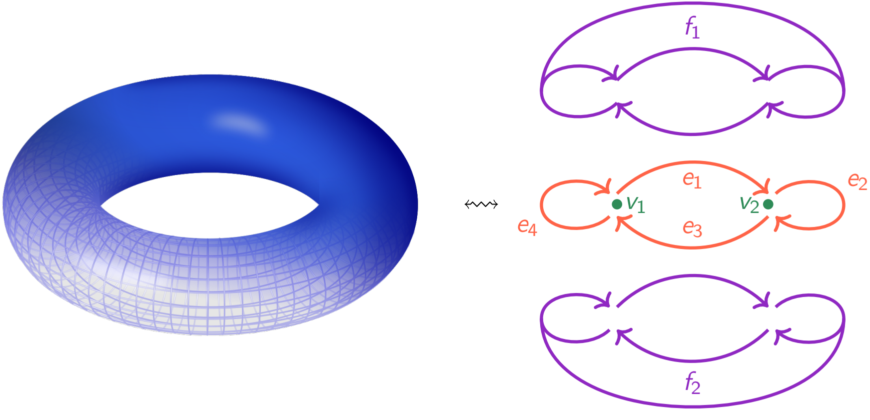
What is...cellular homology?

Or: Winding around

The cell complex – Step 1

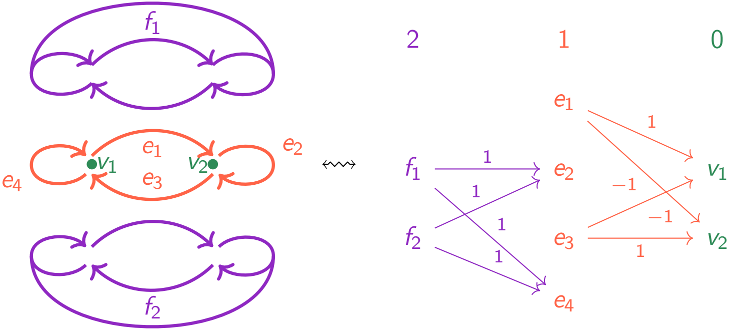


- Given a cell structure we count:

$c_0 = \#\text{vertices} = 2$	$c_1 = \#\text{edges} = 4$	$c_2 = \#\text{faces} = 2$

- Set $C_i = \mathbb{Q}^{c_i}$, with basis being vertices, edges and faces

Walking in circles – Step 2

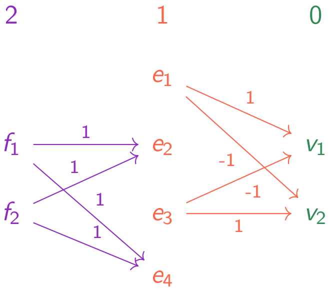


- We construct attaching matrices (after fixing ordered bases):

$$\delta_2 \colon \mathbb{Q}^2 \to \mathbb{Q}^4, \delta_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \delta_1 \colon \mathbb{Q}^4 \to \mathbb{Q}^2, \delta_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

- We have vector spaces $C_i = \mathbb{Q}^{c_i}$ and matrices δ_i

How to count holes? – Step 3



$\Rightarrow$

$$H_0 = \ker(\delta_0)/\mathrm{im}(\delta_1) \cong \mathbb{Q}^1$$

$$H_1 = \ker(\delta_1)/\mathrm{im}(\delta_2) \cong \mathbb{Q}^2$$

$$H_2 = \ker(\delta_2)/\mathrm{im}(\delta_3) \cong \mathbb{Q}^1$$

We get a chain complex:

$$0 \xrightarrow{\;0\;} C_2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}} C_1 \xrightarrow{\begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}} C_0 \xrightarrow{\;0\;} 0$$

kernel 0, rank 0 | kernel 1, rank 1 | kernel 3, rank 1 | kernel 2, rank 0

Take its homology “kernel minus rank”

For completeness: A formal definition

Let X be a cell complex

- The nth cellular chain group is

$$C_n = C_n(X) = \mathbb{Z}\{n\text{-cells}\} = \mathbb{Z}\{e_n^i \mid i \text{ runs over all } n\text{-cells}\}$$

- The nth cellular chain map is

$$\delta_n\colon C_n \to C_{n-1}, \quad \delta_n(\sigma) \text{ is given by the attaching map}$$

- The ith cellular homology is

$$H_n = H_n(X) = \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$$

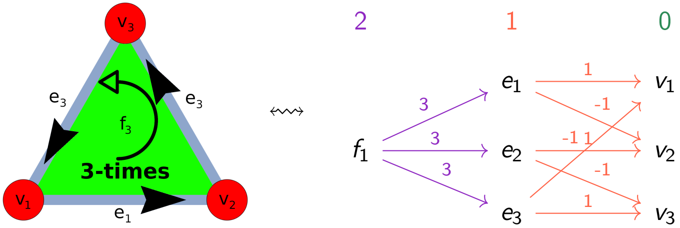
- Cellular homology is a homotopy/homeomorphism invariant

Simplicial/singular homology also exist

Singular homology=simplicial homology=cellular homology for any reasonable X

Singular homology is general, simplicial homology is computable for machines, cellular is computable for humans

Mind the winding maps



- The homology over $\mathbb{Q}$ in this case is $H_0 \cong \mathbb{Q}$, $H_1 \cong 0$, $H_2 \cong 0$

$$0 \xrightarrow{\quad 0 \quad} \mathbb{Q} \xrightarrow{\begin{pmatrix} 3 & 3 & 3 \end{pmatrix}} \mathbb{Q}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}} \mathbb{Q}^3 \xrightarrow{\quad 0 \quad} 0$$

kernel 0, rank 0 | kernel 0, rank 1 | kernel 1, rank 2 | kernel 3, rank 0

- The homology over $\mathbb{Z}$ in this case is $H_0 \cong \mathbb{Z}$, $H_1 \cong \mathbb{Z}/3\mathbb{Z}$, $H_2 \cong 0$

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{\begin{pmatrix} 3 & 3 & 3 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{0} 0$$

Thank you for your attention!

I hope that was of some help.