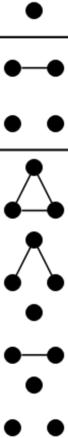
What are...graphs with small spectrum?

Or: ADE is it!

Small PF eigenvalue = few paths



1.1	0
2.1	$1, -1$
2.2	$0, 0$
3.1	$2, -1, -1$
3.2	$\sqrt{2}, 0, -\sqrt{2}$
3.3	$1, 0, -1$
3.4	$0, 0, 0$

4.1	$3, -1, -1, -1$
4.2	$\rho, 0, -1, 1-\rho$
4.3	$2, 0, 0, -2$
4.4	$\theta_1, \theta_2, -1, \theta_3$
4.5	$\sqrt{3}, 0, 0, -\sqrt{3}$
4.6	$\tau, \tau-1, 1-\tau, -\tau$
4.7	$2, 0, -1, -1$
4.8	$\sqrt{2}, 0, 0, -\sqrt{2}$
4.9	$1, 1, -1, -1$
4.10	$1, 0, 0, -1$
4.11	$0, 0, 0, 0$

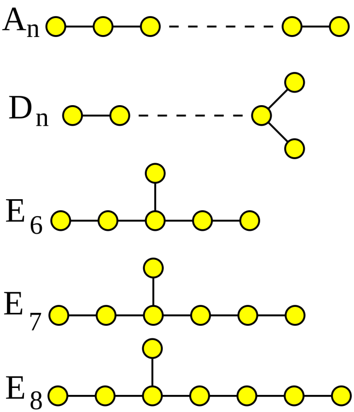
$\tau \approx 1.62$

$\rho \approx 2.56$

$\theta_1 \approx 2.17$

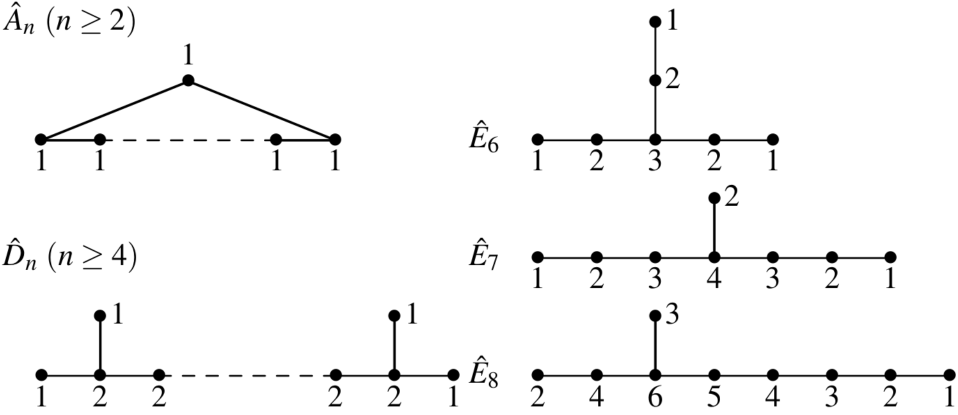
- $PF(G)$ = leading eigenvalue of a graph G
- Recall that number of paths roughly growth like $PF(G)^n$
- Question Can we classify graphs with few paths?

Eigenvalues < 2



- The ADE graphs have $PF(G) < 2$ (easy to see)
- Question Is the list complete?

Eigenvalues ≤ 2



- The affine ADE graphs have $PF(G) = 2$ (easy to see)
- They even have nice PF eigenvectors
- Question Is the list complete?

For completeness: A formal statement

For any connected graph G:

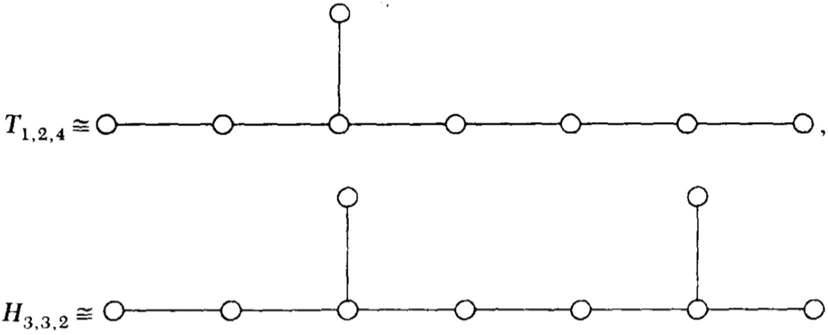
- ▷ The leading eigenvalue $PF(G)$ is < 2 if and only if G is ADE type
- ▷ The leading eigenvalue $PF(G)$ is $= 2$ if and only if G is affine ADE type

- Thus, there are only two infinite families of "few paths graphs"
- A similar statement holds for directed multigraphs
- Here are the eigenvalues of the ADE graphs:

The eigenvalues of A_n are $2\cos i\pi/(n+1)$ $(i = 1,2,\ldots,n)$.
The eigenvalues of D_n are 0 and $2\cos i\pi/(2n-2)$ $(i = 1,3,5,\ldots,2n-3)$.
The eigenvalues of E_6 are $2\cos i\pi/12$ $(i = 1,4,5,7,8,11)$.
The eigenvalues of E_7 are $2\cos i\pi/18$ $(i = 1,5,7,9,11,13,17)$.
The eigenvalues of E_8 are $2\cos i\pi/30$ $(i = 1,7,11,13,17,19,23,29)$.

$2\cos(i\pi/(n+1))$

Going further is difficult



Appear on the list for $PF(G) \in [2, (2+\sqrt{5})^{1/2} \approx 2.058]$

- There are classifications for $PF(G) \in [0, 2+\varepsilon]$ and small enough ε
- The classifications are rather difficult and only go so far
- In general, the set of all $PF(G)$ is not closed in $\mathbb{R}_{\geq 0}$

Thank you for your attention!

I hope that was of some help.