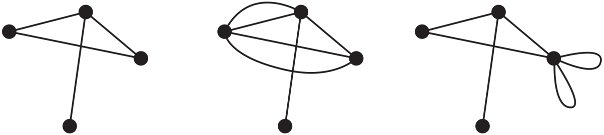
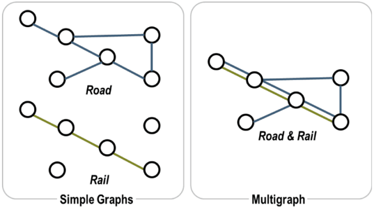
What are...adjacency matrices?

Or: Graphs "=" matrices

Simple graphs and multigraphs

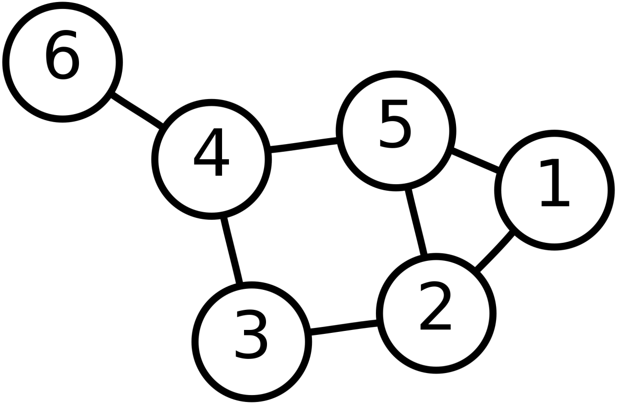


simple graph — *nonsimple graph with multiple edges* — *nonsimple graph with loops*



- Simple graphs "=" vertices+edges with no loops or multiple edges
- Multigraphs "=" vertices+edges with no restriction
- In this series graph = finite simple graph + multigraph = finite multigraph

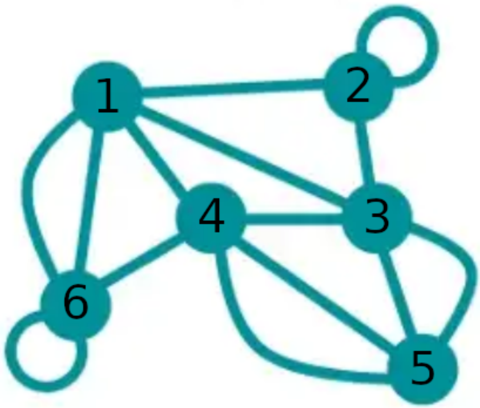
From a graph to a $0,1$ (sym. square diagonal zero) matrix



$$\leftrightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- Underrated fact Graphs "=" $0,1$ matrices = adjacency matrix $A(G)$
- Essentially every vertex corresponds to a column/row, and edges are entries
- Theorem We can go go back-and-forth, *i.e.* there is a bijection $A(\_)$

From a multigraph to a (sym. square) $\mathbb{N}$ matrix



$$\leftrightsquigarrow \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

- Underrated fact Multigraphs "=" $\mathbb{N}$ matrices = adjacency matrix $A(G)$
- Essentially every vertex corresponds to a column/row, and edges are entries
- Theorem We can go go back-and-forth, *i.e.* there is a bijection $A(\_)$

For completeness: A formal statement

The adjacency matrix gives a bijection

$A(\_)\colon \{\text{Multigraphs}\} \to \{\text{symmetric square } \mathbb{N} \text{ matrices}\}$

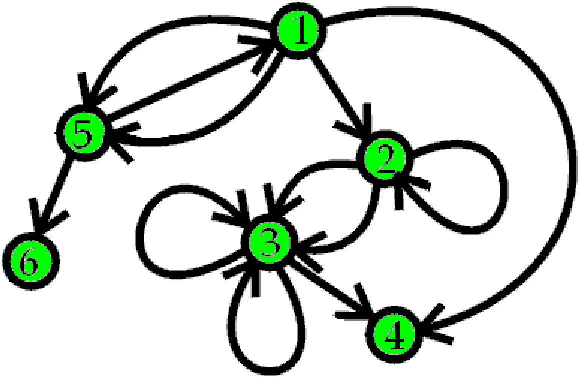
The inverse is the graph obtained by reading the adjacency matrix

- Isomorphism of multigraphs $\leftrightsquigarrow$ conjugation via permutation matrices

$$P_\pi = \begin{bmatrix} \mathbf{e}_{\pi(1)} \\ \mathbf{e}_{\pi(2)} \\ \mathbf{e}_{\pi(3)} \\ \mathbf{e}_{\pi(4)} \\ \mathbf{e}_{\pi(5)} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_4 \\ \mathbf{e}_2 \\ \mathbf{e}_5 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

- Thus, multigraphs "=" $\mathbb{N}$ matrices
- There are similar statements for graphs

From a directed multigraph to a (square) $\mathbb{N}$ matrix



$$\rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- Underrated fact Directed multigraphs "=" $\mathbb{N}$ matrices = adjacency matrix $A(G)$
- Essentially every vertex corresponds to a column/row, and edges are entries
- Theorem We can go go back-and-forth, *i.e.* there is a bijection $A(\_)$

Thank you for your attention!

I hope that was of some help.