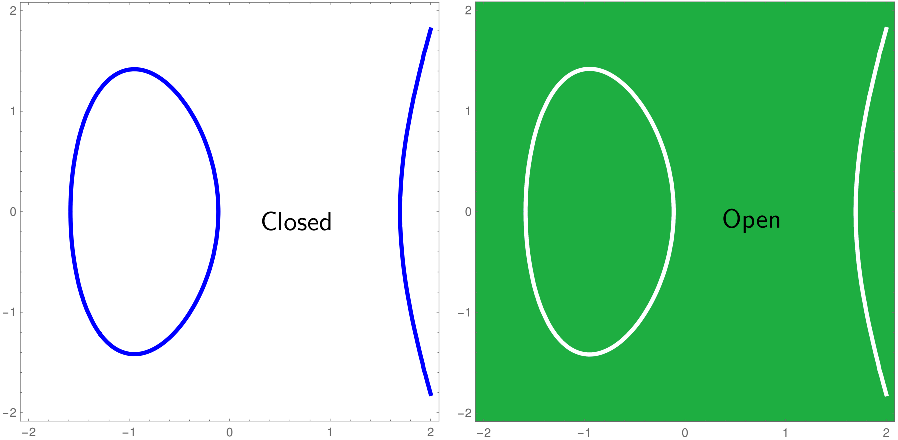
What is...the Zariski topology in algebra?

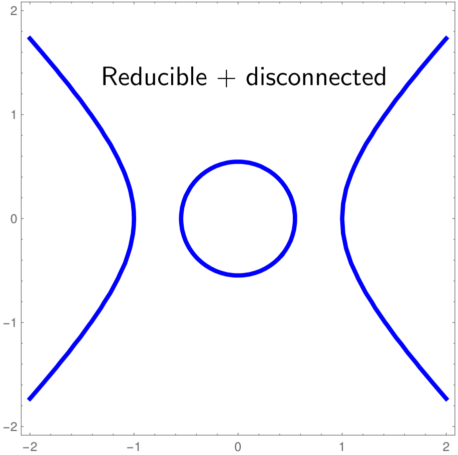
Or: From topology to algebra

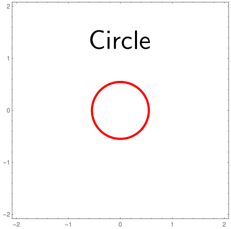
Reminder: Zariski topology

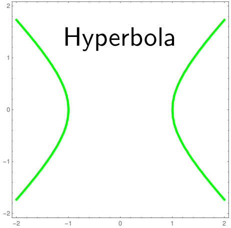


- Zariski topology = closed sets are varieties (small)
- Zariski topology = open sets are complements of varieties (large)
- Today A translation to algebra

Some topological notions make sense...



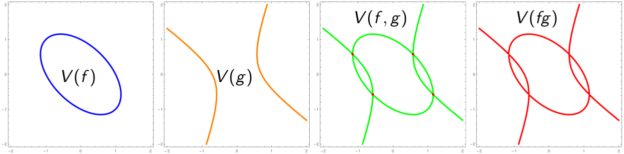




- **Irreducible = opposite of:** A topological space X is reducible if $X = Y \cup Z$ for closed nontrivial spaces Y, Z
- **Connected = opposite of:** A topological space X is disconnected if $X = Y \cup Z$ for closed nontrivial spaces Y, Z with $Y \cap Z = \emptyset$

Back to algebra

Identifying varieties and ideals



- We have bijections

$$\{\text{varieties}\} \overset{1:1}{\longleftrightarrow} \{\text{radical ideals}\}$$
$$X \mapsto I(X)$$
$$V(P) \leftarrow P$$

- Radical ideal means $I = \sqrt{I}$
- One mild catch: the above are order reversing

- Coordinate ring $= \mathbb{K}[V] = \mathbb{K}[x_1, \ldots, x_n]/I(V)$
- The point We pass information between V (geometry) and $\mathbb{K}[V]$ (algebra)
- Question What does the Zariski topology and its properties look like in $\mathbb{K}[V]$?

For completeness: A formal statement

We have translations from geometry to algebra (for $\mathbb{K}$ algebraically closed)

(i) $V = W \cup U$ disconnected $\Rightarrow \mathbb{K}[V] \cong \mathbb{K}[W] \times \mathbb{K}[U]$

(ii) $V \neq \emptyset$ irreducible $\Leftrightarrow \mathbb{K}[V]$ is an integral domain

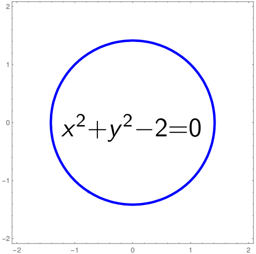
(iii) We have a 1:1 correspondence

$$\{\text{irreducible varieties} \neq \emptyset\} \xleftrightarrow{1:1} \{\text{prime ideals}\}$$

(iv) We have a 1:1 correspondence

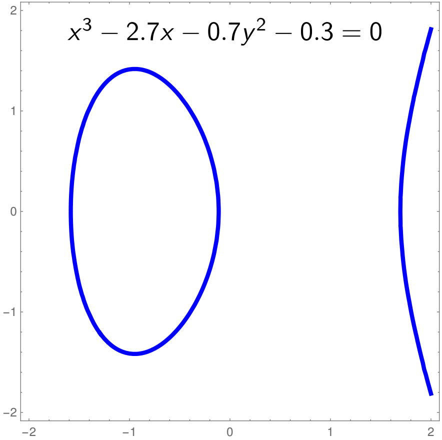
$$\{\text{irreducible components of } V\} \xleftrightarrow{1:1} \{\text{minimal prime ideals in } \mathbb{K}[V]\}$$

(v) More... We will explore this!



$\mapsto (x^2 + y^2 - 2)$ prime ideal

The picture is deceiving...



- The above variety is irreducible
- WTF? The problem is the real picture here: work over $\mathbb{C}$!
- Why? Check that $(x^3-2.7x-0.7y^2-0.3)$ is prime (not too difficult)

Thank you for your attention!

I hope that was of some help.