What are...algebraic varieties?

Or: Zeros!

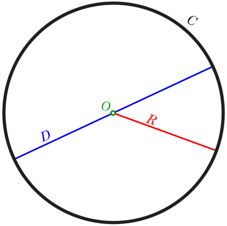
Zero sets

Quadratic Formula

Not too exciting for us: $ax^2 + bx + c = 0$

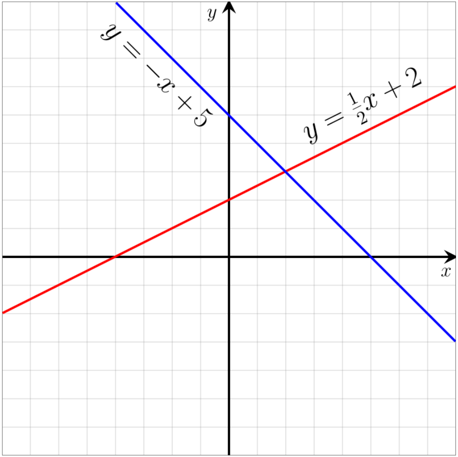
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$x^2 + y^2 - R = 0 \leftrightsquigarrow$



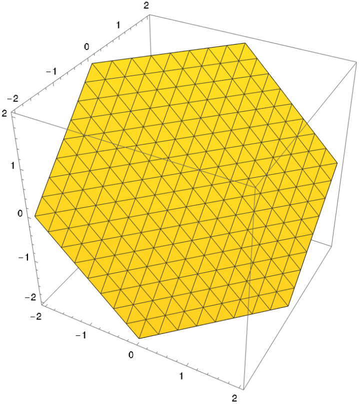
- Algebraic geometry (AG) studies zero sets of polynomials (usually many variables!)
- Not so much of interest in AG are formulas to find roots
- We are rather interested in the shape of these zero sets

Degree one



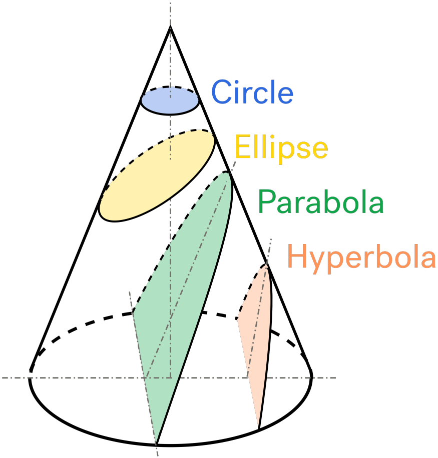
In[5]:= `ContourPlot3D[x - y + z == 0, {x, -2, 2}, {y, -2, 2}, {z, -2, 2}, ContourStyle → Thickness[0.01]]`

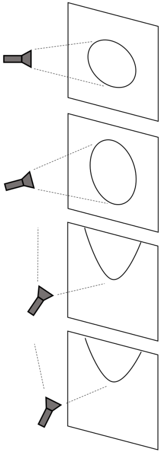
Out[5]=



- Degree of an equation Highest exponent of the appearing variables (taking sums of different variable exponents so that xy^2 is of degree 3)
- Degree zero = constants (ignore), Degree one = linear things (lines, planes, etc.)

Degree two





- Degree two = conic sections
- Example The circle is $x^2 + y^2 - 1 = 0$

For completeness: A formal statement

An affine variety is

$$V = \{v \in \mathbb{K}^n \mid f(v) = 0 \forall f \in P\}$$

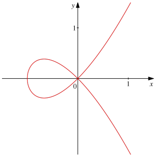
where:

(i) $\mathbb{K}$ is some field

(ii) $P \subset \mathbb{K}[x_1, ..., x_n]$ is a collection of polynomials

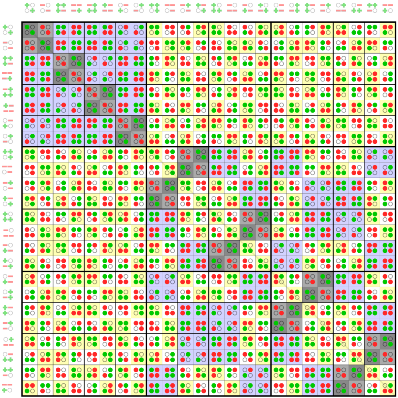
- For $\mathbb{K} = \mathbb{R}$ we can draw nice pictures, but things are a bit ill-behaved

$$y = x^2(x+1) \rightsquigarrow_{\mathbb{R}}$$



- For $\mathbb{K} = \mathbb{C}$ we cannot draw nice pictures, but things are well-behaved
- There are also projective and abstract varieties but let us not worry about them for now

Matrix varieties

$SL_2(\mathbb{F}_3) \leftrightsquigarrow$ 

- Special linear group

$$SL_n(\mathbb{K}) = \{M \text{ a } n\text{x}n \text{ matrix} \mid \det(M) = 1\}$$

- By considering entries as variables, $\det(\_) = 1$ is a polynomial equation
- $SL_n(\mathbb{K})$ is thus an affine variety in $\mathbb{K}^{n^2}$

Thank you for your attention!

I hope that was of some help.