What is...the identity theorem?

Or: Complex versus algebraic geometry

Identity theorem for analytic functions

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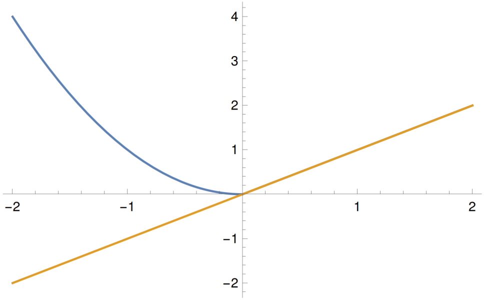
Plot[{Piecewise[{{x^2, x < 0}, {x, x > 0}}],
  Piecewise[{{x, x < 0}, {x, x > 0}}]}, {x, -2, 2}]
```

not determined:



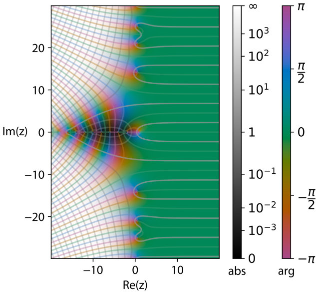
- Analytic function = locally given by a convergent power series (this includes almost all “nice” functions)
- Bonkers These are determined on some nonempty domain (open+connected)
- Example If we know a function is analytic and linear on a disc around the origin (arbitrary small!), then it is linear everywhere

It actually gets better

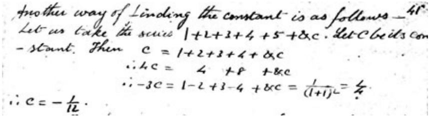
This is used to uniquely extend

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

to all of $\mathbb{C}$

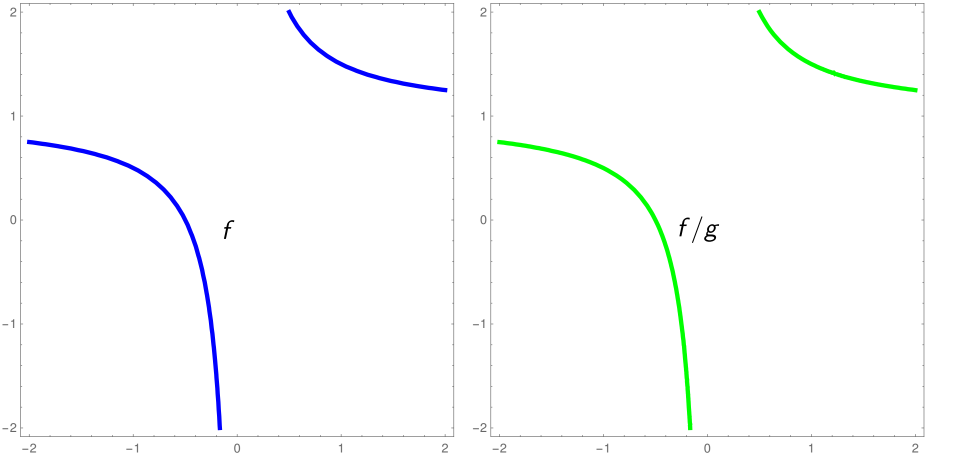


Hence, $\zeta(-1) = -\frac{1}{12}$ "$=$" $1 + 2 + 3 + 4 + \ldots$


Another way of finding the constant is as follows — Let us take the series 1+2+3+4+5+&c. Let C be its constant. Then c = 1+2+3+4+&c
∴ 4c = 4 + 8 + &c
∴ −3c = 1−2+3−4 +&c = $\frac{1}{(1+1)^2}$ = $\frac{1}{4}$
∴ c = $-\frac{1}{12}$.

- Holomorphic roughly means complex differentiable
- In complex analysis holomorphic $=$ analytic, the identity theorem holds
- Holomorphic functions are also called regular functions

Zeros of regular functions



- Recall Varieties = zeros of polynomials $\leftrightsquigarrow$ closed sets
- Better: For $\phi \in \mathcal{O}_V(U)$ the set of zeros of ϕ is also closed
- Proof sketch $\phi = f/g$ so $V(\phi) = V(f)$

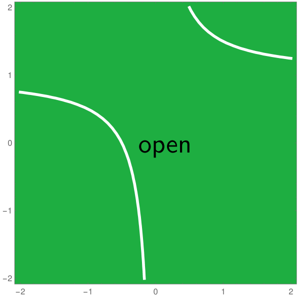
For completeness: A formal statement

$\phi, \psi \in \mathcal{O}_V(U)$ agree on $U' \subset U$ open then

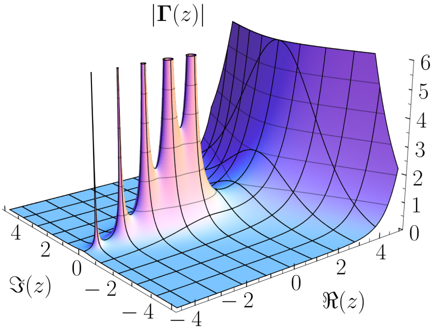
they agree on all of U

This is the identity theorem for regular functions

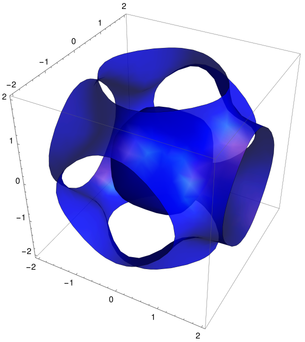
- Here V is an irreducible affine variety
- The main ingredient is the statement on the previous slide
- Not too impressive: open sets in AG are usually very large



Complex versus algebraic geometry



and



- The point is not the theorem itself: its rather "obvious" since open sets in AG are usually very large
- The point I want to drive home is the remarkable analogy between two different theories: complex geometry/analysis and algebraic geometry
- It gets better! We will see several other instances of this along the way

Thank you for your attention!

I hope that was of some help.