What is...the classification of abelian groups?

Or: Factoring numbers.

Can one classify groups? Well...

Almost all groups are not abelian, but it takes a while to get started:

Order	All	Abelian	Ratio
1	1	1	1
2	1	1	1
3	1	1	1
4	2	2	1
5	1	1	1
6	2	1	0.5
7	1	1	1
8	5	3	0.6
9	2	2	1
⋮	⋮	⋮	⋮
256	56092	22	≈0.0004
⋮	⋮	⋮	⋮
2048	Unknown (in 2021)	56	Unknown (in 2021)
⋮	⋮	⋮	⋮

Chinese reminder theorem

There are certain things whose number is unknown. If we count them by threes, we have two left over; by fives, we have three left over; and by sevens, two are left over. How many things are there? Sun-tzu ∼3rd century

$n \equiv 2 \bmod 3$ $n \equiv 3 \bmod 5$ $n \equiv 2 \bmod 7$

Theorem. There is only one solution n between $0 \leq n \leq 3 \cdot 5 \cdot 7$

Caution. You need to have that 3, 5 and 7 are coprime, *e.g.*

$n \equiv 2 \bmod 4$ $n \equiv 2 \bmod 8$

has solutions 2, 10, 18 and 26 below $4 \cdot 8 = 32$

Fundamental examples of abelian groups

The "only" abelian groups are:

Infinite: $\mathbb{Z}$, Finite: $\mathbb{Z}/p^k\mathbb{Z}$, p prime

Chinese reminder theorem gives:

$\mathbb{Z}/1\mathbb{Z}$	1
$\mathbb{Z}/2\mathbb{Z}$	1
$\mathbb{Z}/3\mathbb{Z}$	1
$\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2
$\mathbb{Z}/5\mathbb{Z}$	1
$\mathbb{Z}/6\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, 1 \mapsto (1,1)$	1
$\mathbb{Z}/7\mathbb{Z}$	1
$\mathbb{Z}/8\mathbb{Z} \ncong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	3

For completeness: The formal statement

For any finitely generated abelian group G there exist r and (not necessary distinct) prime powers $q_1, \dots, q_s$ such that

$$G \cong \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{r \text{ copies}} \times \mathbb{Z}/q_1\mathbb{Z} \times \dots \times \mathbb{Z}/q_s\mathbb{Z}$$ Existence

(a) r and $q_1, \dots, q_s$ are (up to renaming) uniquely determined by G Uniqueness

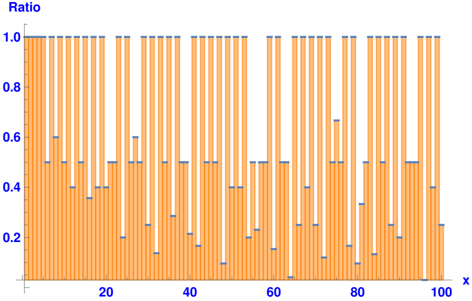
(b) r is called the rank

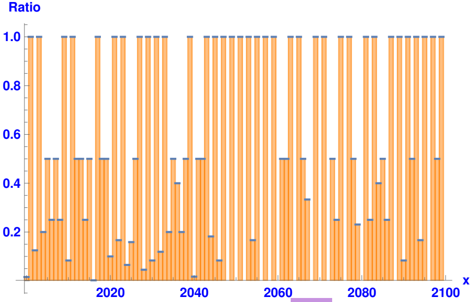
(c) This classifies finitely generated abelian groups

(d) To count their number for a given order is easy Cool!

The finitely generated condition is essential here – for *e.g.* $\mathbb{Q}$ the above fails

Almost all groups are not abelian?





There is no hope to classify all finite groups...

Thank you for your attention!

I hope that was of some help.