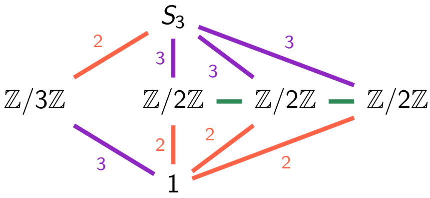
What is...the fundamental theorem of Galois theory?

Or: From roots to groups and back

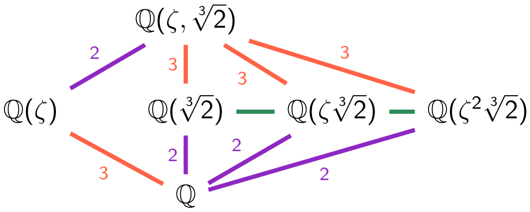
S_3 and its subgroup lattice



- S_3 has six subgroups, four up to conjugacy Green left-right arrow
- A copy of $\mathbb{Z}/3\mathbb{Z}$ of index 2
- Three copies of $\mathbb{Z}/2\mathbb{Z}$ of index 3
- $\mathbb{Z}/3\mathbb{Z}$ is normal, the three copies of $\mathbb{Z}/2\mathbb{Z}$ are not

We have seen this is for roots of polynomials

$\mathbb{Q}(\zeta = e^{2\pi i/3}, \sqrt[3]{2})$ and its subfield lattice



- $\mathbb{Q}(\zeta, \sqrt[3]{2})$ has six subfields, four up $\mathrm{Aut}(\mathbb{Q}(\zeta, \sqrt[3]{2})/\mathbb{Q})$ Green left-right arrow
- A copy of $\mathbb{Q}(\zeta)$ of degree 3
- Three copies of $\mathbb{Q}(\sqrt[3]{2})$ of degree 2
- $\mathbb{Q}(\zeta)$ is Galois over $\mathbb{Q}$, the three copies of $\mathbb{Q}(\sqrt[3]{2})$ are not

Main observation This is the same as for S_3, but upside down

A direct comparison

$$\begin{tikzcd}
 & \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \arrow[dl, dash, "2"'] \arrow[d, dash, "2"'] \arrow[dr, dash, "2"] & & & & \mathbb{Q} \arrow[dl, dash, "2"'] \arrow[d, dash, "2"'] \arrow[dr, dash, "2"] & \\
\mathbb{Z}/2\mathbb{Z} \arrow[dr, dash, "2"'] & \mathbb{Z}/2\mathbb{Z} \arrow[d, dash, "2"'] & \mathbb{Z}/2\mathbb{Z} \arrow[dl, dash, "2"] & \leftrightsquigarrow & \mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "2"'] & \mathbb{Q}(\sqrt{6}) \arrow[d, dash, "2"'] & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash, "2"] \\
 & 1 & & & & \mathbb{Q}(\sqrt{2},\sqrt{3}) &
\end{tikzcd}$$

(a) On the subgroup side this is clear since
$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} = \{(0,0),(1,0),(1,1),(0,1)\}$

(b) On the subfield side this is not so obvious:

- $\mathrm{Aut}(\mathbb{Q}(\sqrt{2},\sqrt{3})) = \{1, f, g, gf\}$ where

$$f = \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \end{cases}, \quad g = \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \end{cases}$$

- The fixed field of f is $\mathbb{Q}(\sqrt{2})$, the fixed field of g is $\mathbb{Q}(\sqrt{3})$ and the fixed field of gf is $\mathbb{Q}(\sqrt{6} = \sqrt{2}\cdot\sqrt{3})$
- There are no other subfields

For completeness: The formal statement

If $\mathbb{L}$ is Galois over $\mathbb{K}$ with Galois group $G(\mathbb{L}/\mathbb{K}) = \mathrm{Aut}(\mathbb{L}/\mathbb{K})$, then:

(a) $\mathbb{L}$ is Galois over any $\mathbb{K} \subset$ subfields $Z \subset \mathbb{L}$

(b) There are inverse bijections, the Galois correspondences:

$$\{\mathbb{K} \subset \text{subfields } Z \subset \mathbb{L}\} \xrightarrow{\cong} \{\text{subgroups of } G(\mathbb{L}/\mathbb{K})\}, Z \mapsto G(\mathbb{L}/Z)$$

$$\{\text{subgroups } H \text{ of } G(\mathbb{L}/\mathbb{K})\} \xrightarrow{\cong} \{\mathbb{K} \subset \text{subfields } Z \subset \mathbb{L}\}, H \mapsto \mathbb{L}^H$$

(c) $(Z \subset Z') \Leftrightarrow (G(\mathbb{L}/Z') \subset G(\mathbb{L}/Z))$ Upside down

(d) $[Z : \mathbb{K}] = |G(\mathbb{L}/\mathbb{K})|/|G(\mathbb{L}/Z)|$ (whenever this make sense) Order and index

(e) (Z is Galois over $\mathbb{K}$) $\Leftrightarrow$ ($G(\mathbb{L}/Z) \triangleleft G(\mathbb{L}/\mathbb{K})$ is normal) Galois and normal

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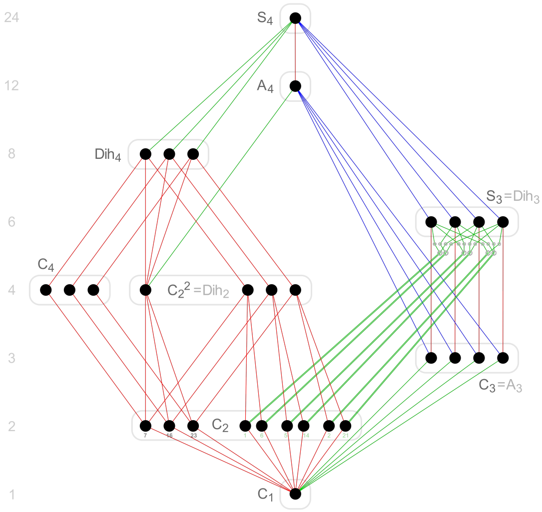
- The Galois correspondence for $\mathbb{L}$ not Galois over $\mathbb{K}$ still works, but is only surjective respectively injective

S_4 vs. explicit roots



- 30 subgroups (black dots), 11 up to conjugacy (gray rectangles – space indicates different conjugacy classes), only A_4, $(C_2)^2$ and 1 are normal
- Homework. Do the same for $\mathbb{Q}$(roots of $X^4 + X + 1$) ;-)

Thank you for your attention!

I hope that was of some help.