What are...minimal polynomials?

Or: Minimal relations

Minimal relations with coefficients in a fixed field $\mathbb{K}$

Relations satisfied by $z = \sqrt{2} + \sqrt[3]{3}$:

- In $\mathbb{Q}(\sqrt{2} + \sqrt[3]{3})$ we have that

$$z - (\sqrt{2} + \sqrt[3]{3}) = 0$$

- In $\mathbb{Q}(\sqrt{2})$ we have that

$$z^3 - 3\sqrt{2} \cdot z^2 + 6 \cdot z - (2\sqrt{2} + 3) = 0$$

- In $\mathbb{Q}(\sqrt[3]{3})$ we have that

$$z^2 - 2\sqrt[3]{3} \cdot z - (2 - \sqrt[3]{3}) = 0$$

- In $\mathbb{Q}$ we have that

$$z^6 - 6 \cdot z^4 - 6 \cdot z^3 + 12 \cdot z^2 - 36 \cdot z + 1 = 0$$

Are these minimal?

Is this well-posed?

- Polynomials annihilating z form an ideal $I_z \subset \mathbb{K}[X]$
- $\mathbb{K}[X]$ is a PID if $\mathbb{K}$ is a field
- Hence, there exists $m_z \in \mathbb{K}[X]$ with

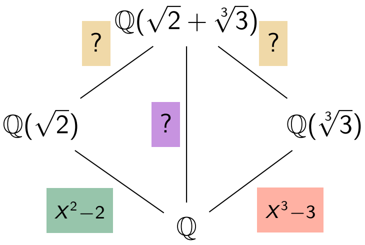
$$I_z = (m_z)$$

- m_z is unique up to scaling

Thus, m_z is the minimal relation in the sense that it is the generator of the annihilating ideal I_z:

The question "What is the minimal relations satisfied by z?" is well-posed

How to prove minimality?



- $X^2 - 2 \in \mathbb{Q}[X]$ is irreducible, so $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ Degree 2
- $X^3 - 3 \in \mathbb{Q}[X]$ is irreducible, so $[\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}] = 2$ Degree 3
- By the tower law $[\mathbb{Q}(\sqrt{2} + \sqrt[3]{3}) : \mathbb{Q}] = 6$ Degree 6
- Thus, $[\mathbb{Q}(\sqrt{2} + \sqrt[3]{3}) : \mathbb{Q}(\sqrt{2})] = 3$ and $[\mathbb{Q}(\sqrt{2} + \sqrt[3]{3}) : \mathbb{Q}(\sqrt[3]{3})] = 2$
  Degrees 3 and 2

In total, the guessed minimal relations are minimal

For completeness: The formal definition/statements

Fields $\mathbb{K} \subset \mathbb{L}$, polynomial ring $\mathbb{K}[X]$, evaluation morphism

$$\mathrm{eval}_z \colon \mathbb{K}[X] \to \mathbb{L}, \quad p(X) \mapsto p(z)$$

The minimal polynomial m_z of $z \in \mathbb{L}$ is defined via $(m_z) = \ker(\mathrm{eval}_z)$ + monic

- This is well-defined up to scaling since $\mathbb{K}[X]$ is a PID Existence
- Is $p \in \mathbb{K}[X]$ irreducible with $p(z) = 0$, then $p = m_z$ up to scaling Uniqueness
- If z is algebraic, then eval_z descents to

$$\overline{\mathrm{eval}}_z \colon \mathbb{K}[X]/(m_z) \xrightarrow{\cong} \mathbb{K}(z)$$

and $1, \ldots, z^{\deg(m_z)-1}$ is a basis of $\mathbb{K}(z)$, so that $[\mathbb{K}(z) : \mathbb{K}] = \deg(m_z)$
This is why we care

Computer talk

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MinimalPolynomial[Root[#^2 - 2 &, 1], X]
-2 + X^2
MinimalPolynomial[Root[#^3 - 3 &, 1], Y]
-3 + Y^3
MinimalPolynomial[Root[#^2 - 2 &, 1] + Root[#^3 - 3 &, 1], Z]
1 - 36 Z + 12 Z^2 - 6 Z^3 - 6 Z^4 + Z^6
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To find the minimal polynomial of $z = x + y$ for $m_x(X) = a_m X^m + \ldots + a_0 = 0$ and $m_y(Y) = b_n Y^n + \ldots + b_0 = 0$ minimal do:

- Compute $m_x(Z - Y)$ Breakfast
- Compute the resultant of $m_x(Z - Y)$ and $m_y(Y)$ Linear algebra

$$e.g.\ \mathrm{Res}_{m_x(Z-Y), m_y(Y)}(Z)$$

- Find smallest factor of $\mathrm{Res}_{m_x(Z-Y), m_y(Y)}(Z)$ Might be tricky

Thank you for your attention!

I hope that was of some help.