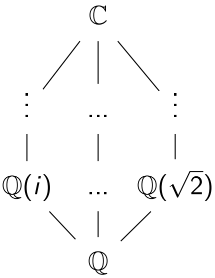
What are...field extensions?

Or: Adjoining roots

A zoo between $\mathbb{Q}$ and $\mathbb{C}$

Question. What are the fields between $\mathbb{Q}$ and $\mathbb{C}$?



- $\mathbb{Q}(i)$ is the minimal field containing $\mathbb{Q}$ and i
- $\mathbb{Q}(\sqrt{2})$ is the minimal field containing $\mathbb{Q}$ and $\sqrt{2}$

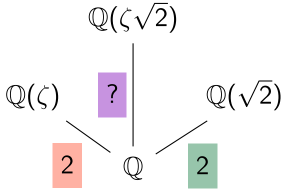
The theory of field extensions asks for a tool box to study this zoo

Minimal relations

Question. What is the minimal relation $\zeta\sqrt{2}$ satisfies in $\mathbb{Q}$?

$(\zeta\sqrt{2})^0$	$(\zeta\sqrt{2})^1$	$(\zeta\sqrt{2})^2$	$(\zeta\sqrt{2})^3$	$(\zeta\sqrt{2})^4$	$(\zeta\sqrt{2})^5$	$(\zeta\sqrt{2})^6$
1	$\zeta\sqrt{2}$	$2\cdot\zeta^2$	$2\cdot\sqrt{2}$	$4\cdot\zeta$	$4\cdot\zeta^2\sqrt{2}$	$8\cdot 1$

- $\sqrt{2}$ satisfies a degree 2 relation $(\sqrt{2})^2 - 2 = 0$ in $\mathbb{Q}$
- ζ satisfies a degree 2 relation $\zeta^2 + \zeta + 1 = 0$ in $\mathbb{Q}$



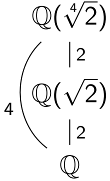
Multiplicative!?

$\mathbb{Q}(\sqrt[4]{2})$ is of dimension 4 over $\mathbb{Q}$:

- 1, $\sqrt[4]{2}$, $(\sqrt[4]{2})^2$ and $(\sqrt[4]{2})^3$ are $\mathbb{Q}$-linear independent
- $(\sqrt[4]{2})^4$ satisfies a relation in $\mathbb{Q}$

$\mathbb{Q}(\sqrt[4]{2})$ is of dimension 2 over $\mathbb{Q}(\sqrt{2})$:

- 1 and $\sqrt[4]{2}$ are $\mathbb{Q}(\sqrt{2})$-linear independent
- $(\sqrt[4]{2})^2$ satisfies a relation in $\mathbb{Q}(\sqrt{2})$



For completeness: The formal definition/statements

A field $\mathbb{L}$, a subfield $\mathbb{K}$ and a subset $M \subset \mathbb{L}$

(a) The field extension $\mathbb{K}(M)$ is the intersection of all fields containing $\mathbb{K} \cup M$
Adjoining roots

(b) The degree $[\mathbb{K}(M) : \mathbb{K}]$ is the dimension of $\mathbb{K}(M)$ as a $\mathbb{K}$ vector space

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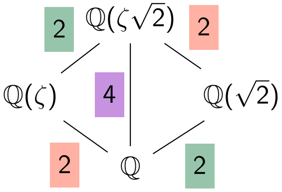
- $[\mathbb{K}(M) : \mathbb{K}] < \infty$ is algebraic, $[\mathbb{K}(M) : \mathbb{K}] = \infty$ is transcendental
- $\mathbb{K}(u) = \mathbb{K}(\{u\})$ are called simple field extensions
- Bases of simple field extensions are given by $1, u, u^2, ..., u^{d-1}$
- (If $f \in \mathbb{K}[X]$ is an irreducible polynomial with $f(u) = 0$, then $[\mathbb{K}(u) : \mathbb{K}] =$ degree of f) $\Rightarrow$ (u satisfies the relation determined by f)
- $[\mathbb{M} : \mathbb{K}] = [\mathbb{M} : \mathbb{L}] \cdot [\mathbb{L} : \mathbb{K}]$ Tower law

The machinery gets going



(a) $\mathbb{Q}(\zeta\sqrt{2})$ is at most of degree 6 $(\zeta\sqrt{2})^6 = 8 \cdot 1$

(b) $\mathbb{Q}(\zeta\sqrt{2})$ contains $\mathbb{Q}(\zeta)$ and $\mathbb{Q}(\sqrt{2})$

(c) The degree of $\mathbb{Q}(\zeta\sqrt{2})$ is divisible by 2

This is not obvious when looking at:

$(\zeta\sqrt{2})^0$	$(\zeta\sqrt{2})^1$	$(\zeta\sqrt{2})^2$	$(\zeta\sqrt{2})^3$	$(\zeta\sqrt{2})^4$	$(\zeta\sqrt{2})^5$	$(\zeta\sqrt{2})^6$
1	$\zeta\sqrt{2}$	$2 \cdot \zeta^2$	$2 \cdot \sqrt{2}$	$4 \cdot \zeta$	$4 \cdot \zeta^2\sqrt{2}$	$8 \cdot 1$

Thank you for your attention!

I hope that was of some help.